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ICOSAHEDRON AND A PAPER DRAGON REVISITED

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Abstract. This is a follow-up paper to the report [R. T. Živaljević and D. R. Živaljević, *Icosahedron and a paper dragon*, The Teaching of Mathematics **28**, 2 (2025), 118–124] on an animated mathematical experiment (simulation) involving the icosahedron. The basic idea of the experiment was to create the simplest possible combinatorial geometric environment, for studying the mathematics behind the morphogenesis of icosahedral shapes in nature. Our objective is to present, in the form accessible to students, teachers, and non-specialists, some of the not so well-known facts about the geometry and combinatorics of the icosahedron, related to this mathematical simulation, emphasising the unity of mathematics and the importance of the multidisciplinary approach in mathematical education.

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1. Introduction

The animation used in this experiment is one of the videos created within the project "Živa matematika" (Math Alive), a project for popularizing mathematics sponsored by the Mathematical Institute SASA (Belgrade) and the Center for the Promotion of Science.

More details about this project can be found in [8]. The author and sole designer of all animations was the second author of [9][1] while the general idea and the overall mathematical expertise were provided by the first author. The original motivation was to produce an attractive animation with rich mathematical content. However, it surpassed the expectation as it evolved [7, 8] into a project, connecting and popularizing methods and ideas from discrete and computational geometry with mathematical applications in biology and chemistry.

1.1. The object of the paper (in a nutshell)

Recall that the *Paper Dragon* [9] (depicted in Figure 1), the main character of the animation "Icosahedron Avatars" [7], is simply a paper strip, divided in twenty numbered, triangular regions. Paper dragon is the initial state of a folding process (metamorphosis) going through several stages, including the stage of an *icosahedron*

[1] Dušan Živaljević (1984–2014)

Figure 1. Paper dragon: the initial state of the animation

and the stage of a *great icosahedron*, see [9] for a detailed description and/or the (self-explanatory) animation [7].

This folding process of the paper dragon resembles, to some extent, the metamorphoses (life cycles) of insects, such as a caterpillar becoming a butterfly, etc. It may be interesting (for a mathematician) to follow this distant biological analogy, especially in light of the fact that protein shells (capsids) of many viruses have the icosahedral form.

It may come as a surprise that, in spite of their different appearances, the icosahedron and the great icosahedron are actually isomorphic, as abstract simplicial complexes (see Corollary 10). This isomorphism is a consequence of a stronger statement (Theorem 9), which exposes a rich algebraic structure behind the beautiful geometry of these mathematical objects.

Theorem 9 is also a basis for another result (Theorem 12) which says that each unfolding of an icosahedron is an unfolding of the associated great icosahedron (and vice versa). As an amusing consequence we obtain that, side by side with the original animation (corresponding to the paper dragon), there are 43380 similar animations, exhibiting metamorphoses of the icosahedron into the great icosahedron, where 43380 is the number of spanning trees in the one skeleton of these polyhedra.

The isomorphism from Corollary 10, expressing the isomorphism of regular polytopes with Schläfly symbols $\{3,5\}$ and $\{3,5/2\}$, is a classical statement, known (in this or dual form $\{5,3\} \leftrightarrow \{5/2,3\}$) to the old masters (Cayley, Möbius, Goursat), see Figures 6-6A and 6-6B in [2, Sections 6.6 and 6.9].

We had a pleasure to rediscover this result ourselves, by looking carefully at the animation [7], many years after its creation. The proof of Theorem 9 is inspired by [5]. Theorem 12, in its present form, doesn't seem to be a well known result.

2. Algebra and geometry of the field $F = \mathbb{Q}[\sqrt{5}]$

In this section we collect algebraic and combinatorial preliminaries, on one hand, and related geometric and topological facts, on the other, needed for application in subsequent sections. One of central objects is the field $F = \mathbb{Q}[\sqrt{5}]$, which allows us to view the icosahedron and the great icosahedron as a pair of *algebraically conjugate polytopes.*

2.1. Buy, borrow, steal, or at least draw a regular icosahedron!

The amusing advise from the title of this subsection is not our own, it is "borrowed" from the article by J. Baez [1], however we have been for many years an admirer and, occasionally, a joyful supporter of this (research) practice.

Following the advise, we continue by borrowing the image of the *icosahedron with three golden rectangles*[2] depicted in Figure 2, which is appropriately referred to as the "golden icosahedron".

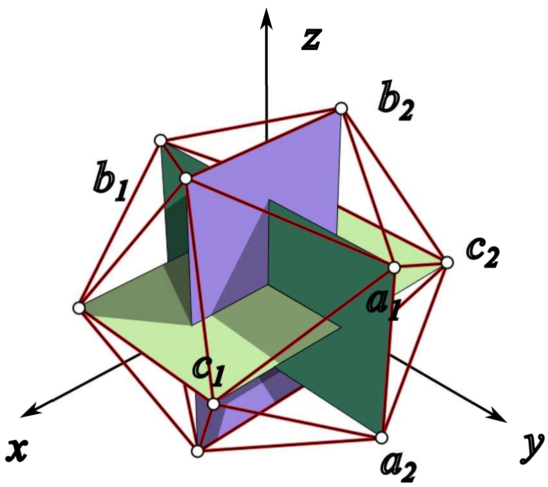

Figure 2. Icosahedron with three golden rectangles

The reasons it is called golden icosahedron is self-evident. The length of the longer (respectively shorter) side of each of the rectangles is 2ϕ (respectively 2), where $\phi = (1+\sqrt{5})/2$ is the golden ratio number.

Twelve vertices of the golden icosahedron are naturally classified into the following three types,

$$(0, \pm\phi, \pm 1),\ (\pm 1, 0, \pm\phi),\ (\pm\phi, \pm 1, 0), \tag{1}$$

each type associated to the corresponding golden rectangle (Figure 2), and characterized by the position of the zero coordinate.

Let us calculate the distances between different pairs of vertices of the golden icosahedron. In reality there is nothing to calculate, since everything is visible and

[2]This file, downloaded from `https://en.wikipedia.org/wiki/File: Icosahedron-golden-rectangles.svg`, is released by the copyright owner into the public domain and included into the Wikimedia Commons freely licensed media file repository.

easily read from Figure 2. Indeed, unless v_1 and v_2 are antipodal, the distance $d(v_1, v_2)$ is either $d_1 = 2$ or $d_2 = 2\phi$, where in the first case v_1 and v_2 are neighbors, while $d(v_1, v_2) = 2\phi$ if and only if v_1 and v_2 are two steps away from each other (measured on the vertex-edge graph of the icosahedron).

Note that there is also a simple combinatorial criterion, allowing us to decide whether $d(v_1, v_2)$ is 2 or 2ϕ without looking at Figure 2 (if we tacitly assume that neither $v_1 = v_2$ nor $v_1 = -v_2$).

1. If v_1 and v_2 are of the same type, say $v_1 = (0, a, b)$ and $v_2 = (0, c, d)$, then $d(v_1, v_2) = 2$ if a and c are of the same sign (and consequently b and d of different sign). Similarly, $d(v_1, v_2) = 2\phi$ in the opposite case when a and c are of the opposite sign (and consequently b and d of the same sign).
2. If $v_1 = (a, b, c)$ and $v_2 = (a', b', c')$ are of different type, say $a = b' = 0$, then we look at the absolute value $|c - c'| \in \{\phi - 1, \phi + 1\}$ and conclude that $d(v_1, v_2)$ is 2 or 2ϕ depending on whether this value is $\phi - 1$ or $\phi + 1$.

For illustration one can easily verify, both geometrically and combinatorially, that the five neighbors of $a_1 = (0, \phi, 1)$ are $\mathcal{N}(a_1) = \{a_2, b_1, b_2, c_1, c_2\}$, where

$$a_2 = (0, \phi, -1),\ b_1 = (+1, 0, \phi),\ b_2 = (-1, 0, \phi),\ c_1 = (+\phi, 1, 0),\ c_2 = (-\phi, 1, 0)\,. \tag{2}$$

2.2. Algebraically conjugate polytopes

We have seen in Section 2.1. that the coordinates of all vertices of the golden icosahedron belong to the field $F = \mathbb{Q}[\sqrt{5}]$. Recall that elements of $\mathbb{Q}[\sqrt{5}]$ are real numbers α which can be expressed as a sum $\alpha = p + q\sqrt{5}$, for some rational numbers $p, q \in \mathbb{Q}$.

An automorphism of a field $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$ is a bijective map $\Phi : F \to F$ which is additive and multiplicative,

$$\Phi(x + y) = \Phi(x) + \Phi(y) \quad \text{and} \quad \Phi(x \cdot y) = \Phi(x) \cdot \Phi(y)$$

for each $x, y \in F$. As a consequence $\Phi(0) = 0$, $\Phi(1) = 1, \ldots, \Phi(r) = r$, for each rational number $r \in \mathbb{Q}$.

Exercise 1. Show that the only non-trivial automorphism of the field $\mathbb{Q}[\sqrt{5}]$ sends $x = p + q\sqrt{5}$ to its *conjugate* $\bar{x} := p - q\sqrt{5}$.

Exercise 2. A regular pentagon is clearly visible in Figure 2. Show that there does not exist a regular pentagon in the plane $\mathbb{R}^2$ such that the coordinates of all its vertices belong to the field $\mathbb{Q}[\sqrt{5}]$.

Each automorphism $\Phi : F \to F$ of a field $F \subseteq \mathbb{R}$ has an associated map $\widehat{\Phi} : F^3 \to F^3$, where $\widehat{\Phi}((x_1, x_2, x_3)) := (\Phi(x_1), \Phi(x_2), \Phi(x_3))$ for each triple $x_1, x_2, x_3 \in F$. The map $\widehat{\Phi}$ is also additive and multiplicative, with respect to the coordinate-wise addition and multiplication. On the other hand the relation $\widehat{\Phi}(\lambda x) = \lambda\widehat{\Phi}(x)$, where $\lambda \in F$ and $x \in F^3$, in general holds only if $\lambda \in \mathbb{Q}$.

Summarising, a rational convex combination $\alpha_1 a_1 + \cdots + \alpha_k a_k$, where $a_i \in F^3$ and $\alpha_i \in \mathbb{Q}$, is mapped to a similar convex combination

$$\widehat{\Phi}(\alpha_1 a_1 + \cdots + \alpha_k a_k) = \alpha_1 \widehat{\Phi}(a_1) + \cdots + \alpha_k \widehat{\Phi}(a_k)\,,$$

the barycenters (with rational weights) of simplices (edges, triangles, tetrahedra), with vertices in F, are mapped to the corresponding barycenters via $\widehat{\Phi}$, etc. This point of view allows us to give a more precise meaning and unambiguous description of *conjugate polytopes* as $\widehat{\Phi}$-transforms of (triangulated) convex polytopes.

Definition 3. Suppose K is a *geometric simplicial complex* [6], or less formally a *"triangulated polyhedron"*, such that all its vertices are in a field F. Then the $\widehat{\Phi}$-transform $\widehat{\Phi}(K)$ of K is another geometric simplicial complex whose geometric simplices are convex hulls (in $\mathbb{R}^3$) of $\widehat{\Phi}$-images $\widehat{\Phi}(L)$ of faces L of K.

Definition 4. (*Conjugate polytope*, cf. [5]). Suppose that all vertices of a convex polytope $Q \subset \mathbb{R}^3$ with triangular facets have coordinates in a subfield $F \subset \mathbb{R}$. Let $\Phi : F \to F$ be an automorphism of F and let $\widehat{\Phi} : F^3 \to F^3$ be the induced map $\widehat{\Phi}((x_1, x_2, x_3)) := (\Phi(x_1), \Phi(x_2), \Phi(x_3))$. Let K be the 2-dimensional, geometric simplicial complex which triangulates the boundary $\partial(Q)$ of Q. Then the "polytope" *conjugate* to Q is the geometric simplicial complex $\widehat{\Phi}(K)$, obtained as the $\widehat{\Phi}$-transform of K in the sense of Definition 3.

Perhaps it should be clarified, in what sense is the geometric simplicial complex $\widehat{\Phi}(K)$, constructed in Definition 4, a genuine polyhedron. This question brings us back to the famous paper (and book chapter [4]), questioning different definitions of a "polytope". It turns out that quite often $\widehat{\Phi}(K)$ is (associated with) a meaningful polytope [5], which deserves to be called a conjugate of Q and denoted by $\widehat{\Phi}(Q)$. This is for example the case (Theorem 9) with the golden icosahedron and its conjugate polytope, the great icosahedron, exhibited side by side in Figure 3.

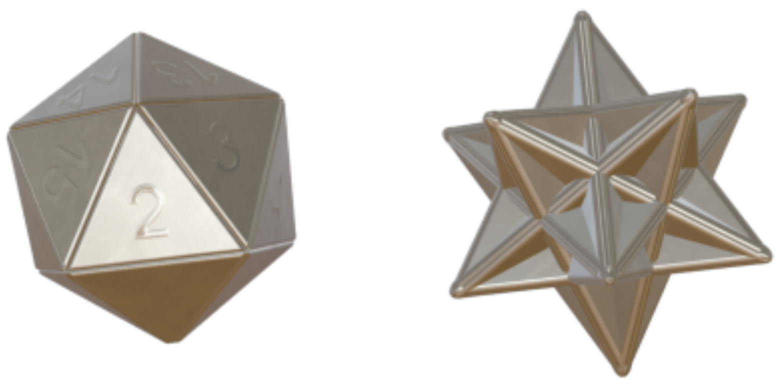

Figure 3. Coiled paper dragons: Icosahedron and the great icosahedron

For the reader's convenience, here is a simple 2D example illustrating the concepts and constructions introduced in this section.

Example 5. Let $P = \mathrm{Conv}(\mathcal{N}(a_1))$ be the pentagon, defined as the convex hull (in $\mathbb{R}^3$) of the set $\mathcal{N}(a_1)$ of five neighbors of a_1, in the golden icosahedron (see Figure 2 and the relation (2)). Let $\Psi : \mathbb{Q}[\sqrt{5}] \to \mathbb{Q}[\sqrt{5}]$ be the only non-trivial automorphism of $\mathbb{Q}[\sqrt{5}]$ (sending $x = p + q\sqrt{5}$ to its *conjugate* $\bar{x} := p - q\sqrt{5}$) and let $\widehat{\Psi}$ be the induced map on $F^3 = \mathbb{Q}[\sqrt{5}]^3$. What can we say about $\widehat{\Psi}(P)$?

First of all we observe that

$$\Psi(\phi) = \Psi\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1-\sqrt{5}}{2} = -\frac{1}{\phi}\,.$$

It follows that $\widehat{\Psi}(a_1) = \widehat{\Psi}((0,\phi,1)) = (0,-\frac{1}{\phi},1) = \frac{1}{\phi}(0,-1,\phi)$. Similarly, $\widehat{\Psi}(a_2) = \frac{1}{\phi}(0,-1,-\phi)$, $\widehat{\Psi}(a_2) = \frac{1}{\phi}(0,-1,-\phi)$, $\widehat{\Psi}(a_2) = \frac{1}{\phi}(0,-1,-\phi)$, $\widehat{\Psi}(a_2) = \frac{1}{\phi}(0,-1,-\phi)$, $\widehat{\Psi}(a_2) = \frac{1}{\phi}(0,-1,-\phi)$.

Exercise 6. Show that the vertices of the pentagon P are mapped by $\widehat{\Psi}$ to the vertices of another golden icosahedron (shrank by a factor of ϕ). What is the image of this pentagon?

The following proposition implies that $\sqrt{5}$ and $-\sqrt{5}$ are indistinguishable (indiscernible) in $\mathbb{Q}[\sqrt{5}]$, in the sense that

$$P(\sqrt{5}) = 0 \Leftrightarrow P(-\sqrt{5}) = 0\,,$$

for each polynomial $P(x)$ with rational coefficients.

Proposition 7. *If $P(x_1, x_2, \ldots, x_n)$ is a multivariate polynomial with rational coefficients, then for each n-tuple $a = (a_1, \ldots, a_n) \in F^n = \mathbb{Q}[\sqrt{5}]^n$,*

$$P(a_1, a_2, \ldots, a_n) = 0 \quad \Leftrightarrow \quad P(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = 0\,.$$

Proof. The proposition is a consequence of the relation $\overline{P}(a_1, a_2, \ldots, a_n) = P(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$. ■

3. Icosahedron and the great icosahedron

Recall that the great icosahedron is usually described as a stellation of the standard icosahedron, see [2, Section 6]. The procedure we outline here (in Exercise 8) is a (more combinatorial) variant of "faceting of platonic solids", described in [2, Section 6].

We already know (Section 2.1) that there are only two numbers, $d_1 = 2$ and $d_2 = 2\phi$, which measure the distance $d(v_1, v_2)$, between two distinct, non-antipodal vertices of the golden icosahedron. The set of all unordered pairs $\{v_1, v_2\}$ of vertices of the golden icosahedron such that $d(v_1, v_2) = 2$, is identified as the set of its edges.

Similarly, the set of all unordered pairs $\{v_1, v_2\}$ of vertices of the golden icosahedron such that $d(v_1, v_2) = 2\phi$, is identified as the set of edges of a graph whose 3-cliques correspond to facets of a triangulation of a sphere.

Exercise 8. Show that, if we allow the self-intersections, the natural geometric realization of this combinatorial sphere is precisely the great icosahedron, depicted in Figure 3 on the right.

Theorem 9. *The great icosahedron (Figure 3) is the conjugate polytope of the golden icosahedron (Figure 2), with respect to the unique non-trivial field automorphism of* $\mathbb{Q}[\sqrt{5}]$.

Proof. Let $\Psi : \mathbb{Q}[\sqrt{5}] \to \mathbb{Q}[\sqrt{5}]$ be the only non-trivial automorphism of $\mathbb{Q}[\sqrt{5}]$ (sending $x = p + q\sqrt{5}$ to its *conjugate* $\bar{x} := p - q\sqrt{5}$) and let $\widehat{\Psi}$ be the induced map on $F^3 = \mathbb{Q}[\sqrt{5}]^3$. Let K be the 2-dimensional geometric simplicial complex, recording the triangulation of the boundary of the golden icosahedron, depicted in Figure 2.

By Definitions 3 and 4 we need to:

(a) determine the image $\mathrm{Vert}_2 := \widehat{\Psi}(\mathrm{Vert}_1)$ where $\mathrm{Vert}_1 = \mathrm{Vert}(Ico_1)$ is the vertex set of the original golden icosahedron Ico_1 (described in (1));

(b) determine the image $\widehat{\Psi}(\{a, b, c\})$ for all triples of vertices $\{a, b, c\} \subset \mathrm{Vert}_1$, corresponding to facets of the golden icosahedron Ico_1.

Since $\Psi(\phi) = -\frac{1}{\phi}$ and $\Psi(r) = r$ for each $r \in \mathbb{Q}$, by applying the map $\widehat{\Psi}$ on vertices listed in table 1, we obtain the following list:

$$\frac{1}{\phi}(0, \mp 1, \pm\phi), \quad \frac{1}{\phi}(\pm\phi, 0, \mp 1), \quad \frac{1}{\phi}(\mp 1, \pm\phi, 0)\,. \tag{3}$$

It follows that the image $\widehat{\Psi}(\mathrm{Vert}_1) = \frac{1}{\phi}\mathrm{Vert}(Ico_2)$ is the vertex-set of another (secondary) golden icosahedron Ico_2, scaled down by the factor ϕ. Since $\widehat{\Psi}$ is an involution, we conclude that the map

$$\phi\widehat{\Psi} : \mathrm{Vert}(Ico_1) \longrightarrow \mathrm{Vert}(Ico_2)$$

is a bijection between vertices of two (different but isometric) copies of the golden icosahedron.

This accounts for (a).

For part (b) it is sufficient to show that for each triple of vertices $\{a, b, c\} \subset \mathrm{Vert}_1$,

$$\|a - b\| = \|b - c\| = \|c - a\| = 2$$
$$\iff \|\phi\widehat{\Psi}(a) - \phi\widehat{\Psi}(b)\| = \|\phi\widehat{\Psi}(b) - \phi\widehat{\Psi}(c)\| = \|\phi\widehat{\Psi}(c) - \phi\widehat{\Psi}(a)\| = 2\phi\,.$$

This follows from Proposition 7, applied to the polynomial $P(x, y) = \|x - y\|^2 - 4$, which implies that

$$\|u - v\|^2 = 2 \Leftrightarrow \|\widehat{\Psi}(u) - \widehat{\Psi}(v)\| = 2 \Leftrightarrow \|\phi\widehat{\Psi}(u) - \phi\widehat{\Psi}(v)\| = 2\phi\,,$$

for each pair of vertices $\{u, v\} \subset \mathrm{Vert}_1$. ■

Corollary 10. *The great icosahedron and the icosahedron are isomorphic as abstract simplicial complexes.*

Indeed, the desired isomorphism is provided by the map $\phi\widehat{\Psi}$.

Exercise 11. Show that the map $\phi\widehat{\Psi} : \mathrm{Vert}(Ico_1) \longrightarrow \mathrm{Vert}(Ico_2)$ is equivariant with respect to the actions of the rotation groups of Ico_1 and Ico_2.

4. Unfolding of the icosahedron and the great icosahedron

If we rewind the paper dragon animation, we obtain a nice example of an *unfolding* of a convex polytope. Recall that we can unfold a 3D polytope by cutting along its edges and laying its connected faces on the 2D plane. When faces do not overlap, the result is called a *net*. The opposite process is naturally referred to as the *folding* (of a planar net) and the paper dragon animation provides a vivid "proof" how a folding of the dragon-net to an icosahedron can be continued to a folding of a great icosahedron.

In order to unfold a convex polytope we must be careful how we cut its boundary. More precisely if $\Gamma = (V, E)$ is the subgraph of the vertex-edge graph of the polytope which records all the cuts, then Γ has no cycles (otherwise we would disconnect the polytope) and each vertex is reached by a cut (otherwise the neighborhood of this vertex would not be flattened).

Summarising, $\Gamma = (V, E)$ produces an unfolding if and only if it is a *spanning tree* in the vertex-edge graph of the polytope. For illustration, each Hamiltonian path on a polytope is its spanning tree and, as explained in [9], this is precisely the origin of our paper dragon.

Theorem 12. *Each unfolding of the icosahedron is also an unfolding of the great icosahedron, and vice versa. In particular, if we choose an arbitrary net of the icosahedron (there are 43380 of them!) and create the corresponding animation, then, sooner or later, we will see the image of the great icosahedron.*

Proof. The proof is surprisingly simple. Let $\Gamma = (V, E)$ be the spanning tree in the vertex-edge graph of the icosahedron Ico, and let $W = W_\Gamma$ be its unfolding associated with the tree $\Gamma = (V, E)$. The unfolding W_Γ is a 2-dimensional simplicial complex (a triangulation of a 2-disc) and, by assumption, there exists a simplicial map $\lambda_\Gamma : W_\Gamma \to Ico$ which is bijective on the interior of the disc W_Γ and a two-fold (2-to-1) covering on the boundary of W_Γ.

By Corollary 10, the boundary simplicial complexes of the icosahedron Ico and of the great icosahedron $GS\text{-}Ico$ are isomorphic (as abstract simplicial complexes). Let $\theta : Ico \to GS\text{-}Ico$ be this isomorphism. Then the composition map $\theta \circ \lambda_\Gamma$ describes the associated unfolding of $GS\text{-}Ico$. ■

4.1. The isomorphism $\phi\widehat{\Psi}$ revisited

The combinatorics of the folding maps $\lambda_\Gamma : W_\Gamma \to Ico$ and $\theta \circ \lambda_\Gamma : W_\Gamma \to GS\text{-}Ico$, introduced in the previous section, provides a new (combinatorial geometric) insight into the isomorphism $\theta = \phi\widehat{\Psi}$ from Corollary 10.

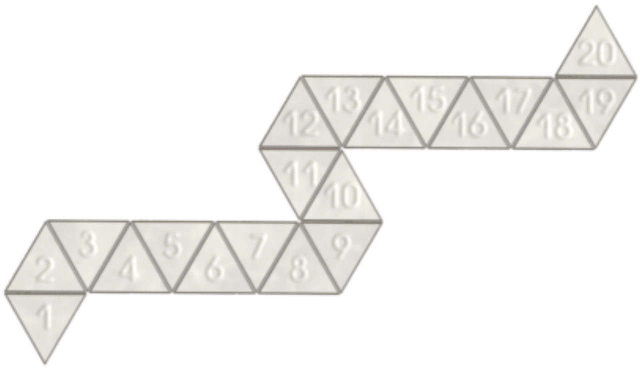

Figure 4. Paper dragon with numbered triangular cells

Let us begin by taking yet another look at the paper dragon (Figure 4), asking ourselves if we can reconstruct the map $\lambda_\Gamma : W_\Gamma \to Ico$ directly from this picture. Indeed, the paper dragon is, for a topologist, just a triangulated topological disc, similar to a regular 22-gon, triangulated by a particular choice of its 19 (pairwise non-intersecting) diagonals. Then (for a topologist) the map λ_Γ is just an identification process, where we glue together pairs of edges of the 22-gon, which should be done in such a way that the result is a 2-sphere.

However, this gluing procedure is by no means unique. Actually, there are Cat_{11} ways to do it, where

$$Cat_n = \frac{1}{n+1}\binom{2n}{n}$$

is the famous *Catalan number*.

Fortunately, in our case we know that the result should be the (boundary of an) icosahedron. In particular each vertex is incident to exactly five triangles and, as a consequence, the free sides of the triangles 7 and 11 should be glued together. After that a new vertex incident to five triangles appears, namely the vertex common to triangles $5, 6, 7, 11$ and 12, the free edges of triangles 5 and 12 should be glued together, etc., a new vertex incident to five triangles emerges, and so on.

Exercise 13. Complete this process, give an explicit description of the map λ_Γ and show that the corresponding spanning tree Γ is a chain of length 11.

What if the result of the gluing process is supposed to be a great icosahedron? The gluing itself should be (topologically) the same, since we know that by Corollary 10 the result should be (topologically and combinatorially) the same.

However, the difference is in geometry. Guided by the animation and/or Example 5 we know that the star of the vertex v, incident to triangles 7, 8, 9, 10 and 11, is the cone over a pentagon (in the case of the icosahedron) and the cone over a pentagram (in the case of the great icosahedron).

This means that, in the case of the great icosahedron, the triangles 7 and 11 are glued together along their free edges e_7 and e_{11} only after e_{11}, the free edge of 11, is rotated counterclockwise around the vertex v, through the angle of 360 degrees, while e_7, the free edge of 7, is not moving at all. In order to obtain the great icosahedron this extra twist should be performed before gluing for each of the vertices of the paper dragon.

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