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ICOSAHEDRON AND A PAPER DRAGON

Rade T. Živaljević and Dušan R. Živaljević

Abstract. We report on a mathematical video experiment involving the icosahedron. The animation created for this experiment was originally designed for the "Živa matematika" (Math Alive) project sponsored by the Mathematical Institute SASA (Belgrade) and the Belgrade Center for Promotion of Science. One of the motivations for the experiment was to create a simple, combinatorial geometric environment, involving a sequential blueprint that generates an icosahedron, in hope that this may eventually shed some new light on the mathematics behind the morphogenesis of icosahedral shapes in nature.

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Key words and phrases: Icosahedron; Hamiltonian path.

1. Introduction

Make a triangulated tape ("paper dragon") following the pattern depicted in Figure 1. Start bending the tape simultaneously at each edge connecting two adjacent triangles.

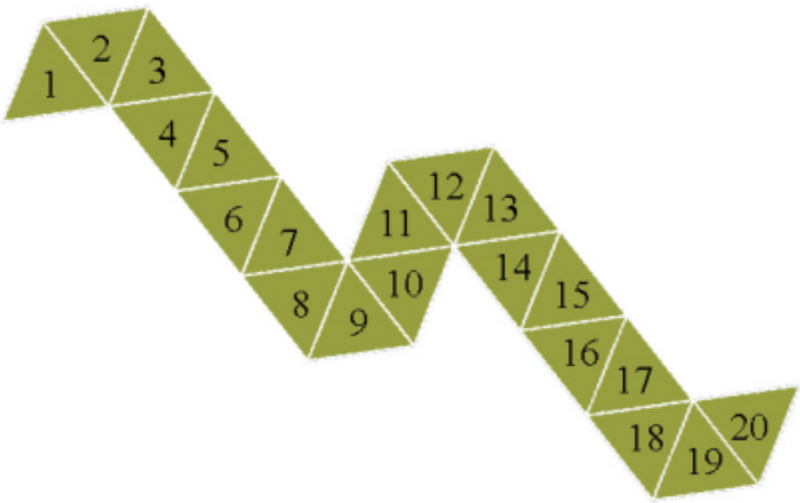

Figure 1

More precisely, choose an angle α between 0 and 180 degrees and fold the tape along the edge connecting triangles 1 and 2 so that the associated dihedral angle is equal to α. Continue this process by folding the tape along the common edge of triangles 2 and 3 so that the dihedral angle determined by triangles 2 and 3 is equal to α, etc. It is assumed that folding is always on the same side, i.e., that the edge forms a ridge from the reader's point of view. We stop when each of the 19 dihedral angles between adjacent triangles is equal to α.

Definition 1. The geometric shape (figure) obtained by the process of folding the triangulated tape depicted in Figure 1 along each of the edges connecting the consecutive triangles for a given angle α is denoted by $D[\alpha]$ and referred to as *α-paper dragon* or *α-dragon* for short.

The π-dragon $D[180°] = D[\pi]$ is by construction (Definition 1) precisely the "paper dragon" depicted in Figure 1. The 0-dragon $D[0]$, arising as the opposite boundary case when $\alpha = 0°$ is the case when the tape folds down to a triangle.

Question 2. What happens with the triangulated tape when the dihedral angle α is in between these two extremal cases?

Problem 3. The general problem we study in this paper is the mathematical structure of objects $D[\alpha]$. For this reason we allow self-intersections, i.e., the triangles freely move during the bending process uninterrupted by other triangles. We want to understand what are the most interesting shapes among the objects $D[\alpha]$ and how are they changing when the dihedral angle α is slowly decreasing from 180° to 0°. We want to determine how different objects $D[\alpha]$ are related to each other by tracking the numbering of the triangles during the different phases of the bending process.

2. Animation of the folding process

Figure 2. $D[\alpha]$ for $\alpha = 180°$

The reader is strongly recommended to visit the site
`http://www.rade-zivaljevic.appspot.com/`,

as well as other addresses listed in [4], with links to thematic mathematical animations created within the project "Živa matematika" (Math Alive). One of these animations (*Icosahedron Avatars*) displays the metamorphosis of the object $D[\alpha]$ when α changes from 180° to 0°. For the reader's convenience here we reproduce some of the most interesting images (snapshots) from this animation.

Figure 2 essentially reproduces the original triangulated tape, that is the object $D[\pi]$. The next image (Figure 3, left) depicts the object $D[\alpha_1]$ where (approximately) $\alpha_1 = 150.05°$. This is the first moment when the triangulated tape (apparently) makes a closed form (the paper dragon bites its tail).

QUESTION 4. Determine the exact value of the angle α_1. Can it be a rational multiple of π? How do we know that triangles 1 and 20 perfectly fit together?

Figure 3. $D[\alpha]$ for $\alpha_1 \approx 150.05°$ and $\alpha_2 \approx 138.19°$

The next, and the one of the most interesting phases in the bending process is when the dihedral angle reaches the value $\alpha_2 \approx 138.19°$ (Figure 3, right). The triangulated tape has formed an *icosahedron* with all 20 triangles in the role of faces. This is not really a surprise since the shape of the "paper dragon" depicted in Figure 1 was carefully chosen precisely to achieve this goal. For this purpose we used a Hamilton cycle of the graph of edges and vertices of the dodecahedron which is dual to the icosahedron depicted in Figure 3, right (see the following section for the details).

However, the bending process is continued. There is no apparent reason why there should be any regular or semi-regular form emerging from the "cacophony" of moving triangles and corresponding dihedral angles. Nevertheless, for $\alpha_3 \approx 109.47°$ and $\alpha_4 \approx 70.53°$ two new regular forms suddenly emerge, the octahedron and the tetrahedron, Figure 4.

Even greater surprise awaits the observer when the dihedral angle reaches the value $\alpha_5 \approx 41.81°$. A wonderful shape emerges, Figure 5. This is the so-called *great (stellated) icosahedron*, one of four Kepler-Poinsot polyhedra (nonconvex regular polyhedra), `http://en.wikipedia.org/wiki/Great_icosahedron.` It can be also described as one of icosahedron stellations.

Figure 4. $D[\alpha]$ for $\alpha \approx 109.47°$ and $\alpha \approx 70.53°$

Figure 5. $D[\alpha]$ for $\alpha_5 \approx 41.81°$ and $\alpha_6 = 0°$

After passing through the great icosahedron stage there are no more surprises. The paper dragon curls into a triangular form (and "falls asleep").

3. The origin of the paper dragon

As already mentioned in the previous section, the shape of the paper dragon depicted in Figure 1 was carefully chosen. We want to guarantee in advance that an icosahedron must appear as one of the shapes $D[\alpha]$ for an appropriate angle α. This is achieved by designing a path on the icosahedron that visits each face exactly once and returns to the beginning position (a closed Hamiltonian path).

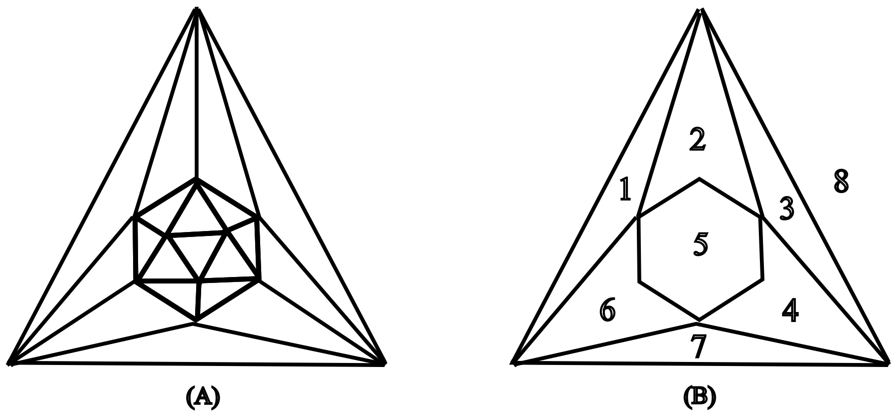

Figure 6. The plan of an icosahedral castle

Figure 6 (A) is an accurate representation of mutual positions of faces of an icosahedron (*Schlegel diagram*) so the problem of designing such a path can be informally rephrased as follows. Suppose that Figure 6 (A) represents a plan of a castle which has 19 triangular rooms. Assume that there is a door at each wall; in particular one can go from each of the rooms to any of the adjacent rooms. Also one can enter the castle through the gates leading to rooms labeled by 1, 3 and 7. In order to describe (or memorize) a closed path visiting each room exactly once, we separate the rooms into 7 apartments (Figure 6 (B)) and design a closed path visiting each of these apartments exactly once.

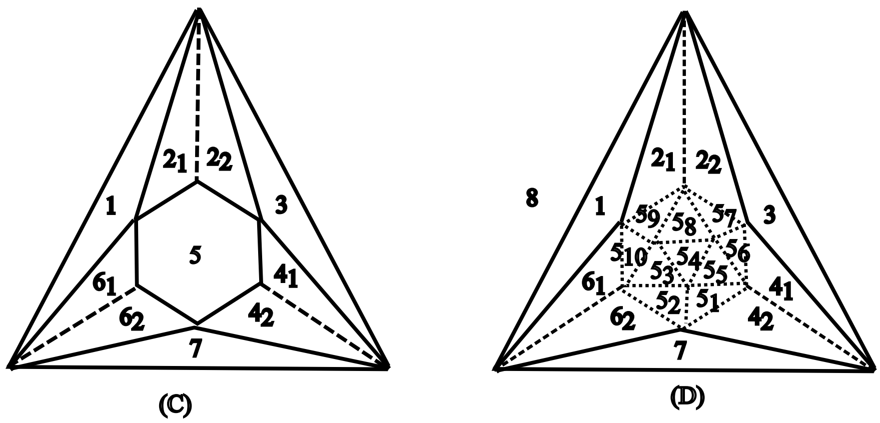

Figure 7. Hamiltonian path of the icosahedron

After that it is not difficult to see how this path can be refined to an actual path visiting each of the 19 rooms (Figure 7 (C)). Observe that when we enter a room, say we move from room 4 to the room labeled by 5_1 (Figure 7 (D)) then there are two possibilities how to continue. We can either move to the left (in our example to the room 5_2) or to the right (to the room labeled by 5_5). Consequently, the word recording the left-right movements of the path (assuming 1 is the initial position) is the following

L L L R L R L R R R L L L R L R L R R

Interchanging the letters L and R we obtain another word describing a Hamiltonian path on the icosahedron and this is precisely the code used to design the paper dragon depicted in Figure 1 (see Figure 8).

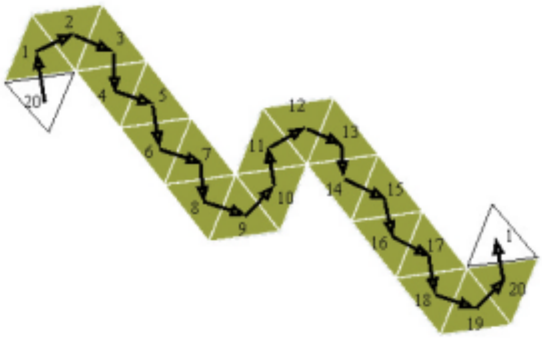

Figure 8. RRRLRLRLLLRRRLRLRLL

4. Motivation and further research

The animation used in this experiment was one of the videos created within the project "Živa matematika" (Math Alive), a project for popularizing mathematics sponsored by the Mathematical Institute (SASA) and Belgrade Center for the Promotion of Science. For more details about this project see `http://www.rade-zivaljevic.appspot.com/`. The sole designer of all animations was the second author while the general idea and the overall mathematical expertise were provided by the first author. The original motivation was to produce an attractive animation with rich mathematical content. However, it turned out to be a potentially interesting project in itself connecting problems of discrete and computational geometry with mathematical applications in biology and chemistry.

The references [1] and [2] address questions about regular polyhedra which are potentially amenable to the experimental approach utilized in our paper. The reference [3] illustrates the relevance of the question of studying combinatorial geometric blueprints, based on Hamiltonian paths, for the morphogenesis of icosahedra in viral *capsids* and related structures.

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R. T. Živaljević and D. R. Živaljević, *Icosahedron and a paper dragon*, Proceedings of the First Mathematical Conference of Republic of Srpska, Pale, May 21–22, 2012.

It is reprinted in this issue of The Teaching of Mathematics, with kind permission of the publisher, as an introduction to the paper:

R. T. Živaljević, *Icosahedron and a paper dragon revisited.*

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3. `http://www.youtube.com/watch?v=IGmjTyAQccQ`
4. `https://www.youtube.com/watch?v=9gRx66xKXek`
5. `https://www.youtube.com/watch?v=rSdMvkY3NAc`

R.T.Ž.: Mathematical Institute of the Serbian Academy of Sciences and Arts
ORCID: 0000-0001-9801-8839
E-mail: `rade@mi.sanu.ac.rs`

D.R.Ž.: Freelance artist, Belgrade

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