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FROM FEYNMAN'S TRIANGLE TO FEYNMAN'S TETRAHEDRA

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Abstract. We investigate a natural extension of so called planar t-Feynman configurations, referring to triangles, to three dimensional t-Feynman configurations, referring to tetrahedra. Our main result extends to three dimensions the well-known Routh's formula for planar t-Feynman configurations.

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1. Introduction

If each vertex of a triangle ABC is joined by a "cevian" to the point $1/3$ along the opposite side (measured say anti-clockwise), then the area ABC is 7 times the area of the triangle determined by the cevians.

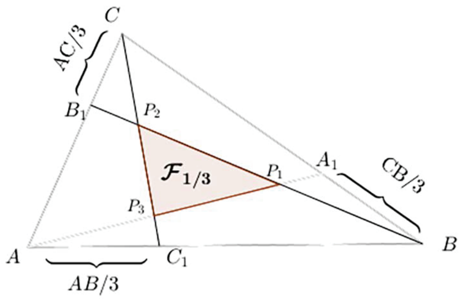

Fig. 1. The classical Feynman's configuration. The area of ABC is seven times the area of $\mathcal{F}_{1/3}$.

The result shown in Fig. 1 is often referred to as *Feynman's theorem* and the central triangle is the classical *Feynman's triangle*. It appears that the great physicist tried to show the theorem at the end of a dinner with a guest, Prof. Kai Li Chung of Stanford University during a visit to Cornell University. Feynman proved the theorem for equilateral triangles. In the planar case, the most natural extensions are those obtained replacing $1/3$ with a positive parameter $t < 1$. Examples of *t-Feynman configurations* are given below (see Fig. 2).

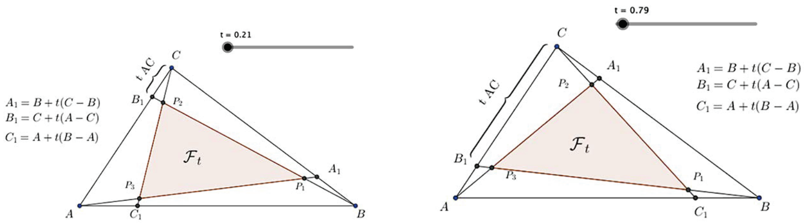

Fig. 2. The 0.21-Feynman and the 0.79-Feynman configurations

The ratio between ABC and its t-*Feynman triangle* is given by the following Routh's Formula (see [4, Eq. 13.55]).

$$\frac{ABC}{\mathcal{F}_t} = \frac{[(\frac{1-t}{t})^2 + \frac{1-t}{t} + 1]^3}{((\frac{1-t}{t})^3 - 1)^2} = \frac{[(\frac{1-t}{t})^2 + \frac{1-t}{t} + 1]}{((\frac{1-t}{t}) - 1)^2} = \frac{(t^2 - t + 1)}{(2t - 1)^2}. \tag{1}$$

More recently, some connections with elementary number theory have been highlighted (see [2, 3, 9, 10, 11] and references therein).

The question of extending the t-Feynman's configurations to tetrahedra appears naturally. In Section 2, for every $t \in (0, 1)$ we give the corresponding construction of the t-Feynman's configurations for tetrahedra. It turns out that every tetrahedron has six distinct configurations of t-Feynman's tetrahedra which have the same volume. The main theorem is in Section 3, where we determine the ratio $R(t)$ between the original tetrahedron and one of its "t-Feynman's tetrahedra" (see Theorem 3.1). As a special case, when $t = 1/3$, it turns out that the ratio is 15 (see Corollary 4.1). This is the corresponding result in the Euclidean space of the classical Feynman's theorem (1).

In Section 4 we highlight that there are infinitely many rational parameters t such that $R(t)$ is an integer (Corollary 4.2). In particular, replacing t with a suitable integer parameter u, we will obtain the notable sequence A006003 of OEIS (see Remark 4.3).

2. The t-Feynman's tetrahedra

First of all, we note that as well as in the Euclidean plane, in the Euclidean space the affine transformations also preserve the parallelism between lines and between planes and also the relationships between volumes, between the areas of coplanar figures and the lengths between collinear segments (see [4]). Therefore, in order to determine the ratio between the volume of a tetrahedron $\mathcal{T}$ and its "t-Feynman's tetrahedra", we may start from an arbitrary tetrahedron.

Let $\mathcal{T} = \{A_0, B_0, C_0, D_0\}$ be a tetrahedron represented as in Fig. 3 and note that there are three *skew quadrilaterals associated with* $\mathcal{T}$: $\mathcal{Q}_1$, Q_2 and $\mathcal{Q}_3$.

Clearly each $\mathcal{Q}_i$ may be ordered in two ways by choosing an ordered 4-tuple and its opposite order. In this way we have six *ordered skew quadrilaterals associated*

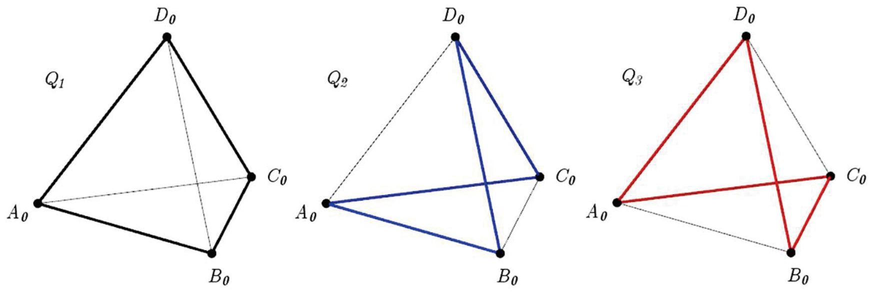

Fig. 3. The skew quadrilaterals associated with a regular tetrahedron: black, blue and red

with $\mathcal{T}$:

$$\mathcal{Q}'_1 = (A_0, B_0, C_0, D_0), \qquad \mathcal{Q}''_1 = (A_0, D_0, C_0, B_0);$$
$$\mathcal{Q}'_2 = (A_0, C_0, D_0, B_0), \qquad \mathcal{Q}''_2 = (A_0, B_0, D_0, C_0);$$
$$\mathcal{Q}'_3 = (A_0, D_0, B_0, C_0), \qquad \mathcal{Q}''_3 = (A_0, C_0, B_0, D_0).$$

We start by focusing our attention on the ordered quadrilateral $\mathcal{Q}'_1$. Note that its sides are: A_0B_0, B_0C_0, C_0D_0, D_0A_0.

Taking as a model the standard construction in 2-dimensional case, here we will construct the t-Feynman's tetrahedron of $\mathcal{T}$ (referred to $\mathcal{Q}'_1$), say $\mathcal{T}_{\mathcal{Q}'_1}$, by means of 4 *cevian planes*. We consider (orderly) the t-parts A_1A_0, B_1B_0, C_1C_0, D_1D_0, of the sides A_0B_0, B_0C_0, C_0D_0, D_0A_0 (see Fig. 4). Using the parametric equations for segments, we have

$$A_1 := (1-t)A_0 + tB_0, \qquad B_1 := (1-t)B_0 + tC_0,$$
$$C_1 := (1-t)C_0 + tD_0, \qquad D_1 := (1-t)D_0 + tA_0.$$

Now, we will denote by π_{C_1}, the plane through C_1 and the opposite side A_0B_0 (see the red plane in Fig. 4, left and right), and in a similar way we may consider the planes π_{A_1},π_{B_1},π_{D_1}, (see π_{A_1}-blue and π_{D_1}-green, Fig. 4, right).

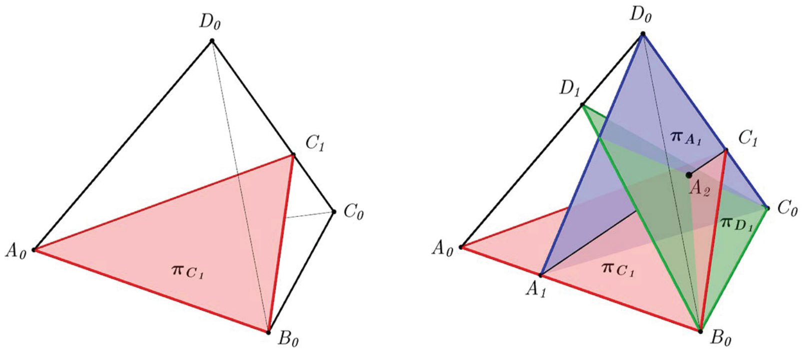

Fig. 4. The construction of a vertex of the t-Feynman's tetrahedron of a regular tetrahedron when $t = 1/3$

Clearly, the intersection of three planes among π_{A_1}, π_{C_1}, π_{B_1}, π_{D_1} determines one point. In this way, we may consider four points A_2, B_2, C_2, D_2 as the intersections of the four choices of such triples (see Fig. 5). This construction depends on the parameter t, thus it is natural to define $\mathcal{T}_{\mathcal{Q}'_1}(t) = \mathcal{T}(A_2, B_2, C_2, D_2)$ as *the* t*-Feynman's tetrahedron of* $\mathcal{T}$ *with respect to* $\mathcal{Q}'_1$. In a similar way, we may construct $\mathcal{T}_{\mathcal{Q}''_1}(t)$, $\mathcal{T}_{\mathcal{Q}'_2}(t)$, $\mathcal{T}_{\mathcal{Q}''_2}(t)$, $\mathcal{T}_{\mathcal{Q}'_3}(t)$, and $\mathcal{T}_{\mathcal{Q}''_3}(t)$. When there are no ambiguities, we use the notation $\mathcal{T}_{\mathcal{Q}}(t)$.

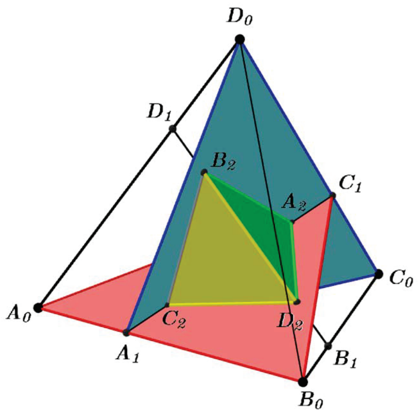

Fig. 5. The vertices of the t-Feynman tetrahedron of $\mathcal{T}$ with respect to Q'_1

REMARK 2.1 We note that, traversing the quadrilateral $\mathcal{Q}_1$ in two opposite directions is equivalent to replacing the parameter t with its complement $(1-t)$, for every $t \in (0,1)$. Thus, by the above construction, the tetrahedra $\mathcal{T}_{\mathcal{Q}'_1}(t)$ and $\mathcal{T}_{\mathcal{Q}''_1}(1-t)$ are congruent.

REMARK 2.2. Given a regular tetrahedron $\mathcal{T}$ and chosen a parameter t, one may expect each t-Feynman's tetrahedron to be regular. The following example shows that it is not true. Indeed, the four faces of each Feynman's tetrahedron are usually isosceles but not equilateral triangles. This will be proved more generally in Theorem 3.4. Here, we give an example.

EXAMPLE 2.3. Let $\mathcal{T} = A_0B_0C_0D_0$ be a regular tetrahedron whose sides length is 1, and consider the ordered skew quadrilateral $\mathcal{Q}'_1 = (A_0, B_0, C_0, D_0)$. After the construction of the Feynman's tetrahedron $\mathcal{T}_{\mathcal{Q}'}(1/3)$ of $\mathcal{T}$, we observe that the measure of its sides is not the same (see Fig. 6). More precisely, at the third decimal digit we have:
$A_2B_2 = 0.39441$, $B_2C_2 = 0.39441$, $C_2D_2 = 0.39441$, $D_2A_2 = 0.39441$ (red sides);
$A_2C_2 = 0.44721$, $B_2D_2 = 0.44721$ (blue sides).

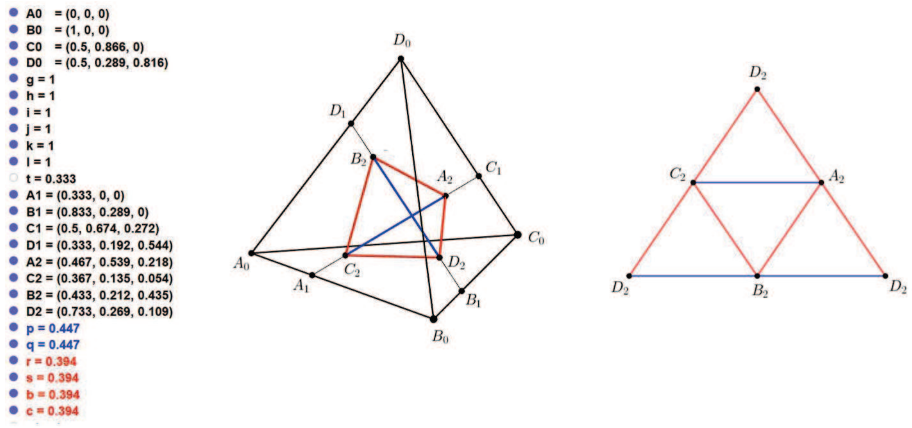

Fig. 6. In the middle, the unit regular tetrahedron $\mathcal{T} = \{A_0, B_0, C_0, D_0\}$ and its 1/3-Feynman's tetrahedron $\mathcal{T}_{\mathcal{Q}_1'}(1/3)$, (by Geogebra). Note that $p = A_2C_2$ and $q = B_2D_2$ (blue sides), are different from $r = A_2B_2$, $s = B_2C_2$, $b = C_2D_2$ and $c = D_2A_2$ (red sides). The right side of the figure shows the development of Feynman's tetrahedron, as four isosceles (non-equilateral) triangles.

3. The main result

THEOREM 3.1. *Let $\mathcal{T}$ be a tetrahedron, and let $\mathcal{Q}$ be an associated ordered skew quadrilateral of $\mathcal{T}$. Then for every positive real number $t < 1$, we have:*

$$\text{(2)}\qquad \frac{\operatorname{vol}(\mathcal{T})}{\operatorname{vol}(\mathcal{T}_{\mathcal{Q}}(t))} = \left| \frac{t^2 + (1-t)^2}{(1-2t)^3} \right|,$$

where $\mathcal{T}_{\mathcal{Q}}(t)$ is the t-Feynman tetrahedron of $\mathcal{T}$ with respect to $\mathcal{Q}$. Moreover, $\operatorname{vol}(\mathcal{T}_{\mathcal{Q}}(t)) = \operatorname{vol}(\mathcal{T}_{\mathcal{Q}}(1-t))$.

Proof. As the tetrahedra are affinity equivalent, without the loss of generality, we may choose $\mathcal{T}$ whose vertices are:

$$\text{(3)}\quad A_0 = (1,0,0);\ \ B_0 = \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\Big);\ \ C_0 = \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\Big);\ \ D_0 = (0,0,\sqrt{2}).$$

Thus $\mathcal{T} = \{A_0, B_0, C_0, D_0\}$ is a regular tetrahedron, whose sides lenghts is $L = \sqrt{3}$. In particular:

$$\operatorname{vol}(\mathcal{T}) = \frac{\sqrt{2}}{12} L^3 = \frac{\sqrt{6}}{4}.$$

We will focus on the ordered skew quadrilateral $\mathcal{Q}_1' = (A_0B_0C_0D_0)$.

The computation of the volume of the t-Feynman's tetrahedron $\mathcal{T}_{\mathcal{Q}_1'}(t)$ involves the parameter t, and as we will see it is of certain complexity. To simplify the calculus, we will use symbolic computation software that allows us to define suitable functions.

We start from $\mathcal{T}$, whose vertices are given by (3). First, we define the function $\mathcal{F}(t)$, which is the parametric equation of the segment AB:

$$\mathcal{F}(A,B,t) = (1-t)A + tB. \tag{4}$$

Using the function (4), and the coordinates (3) of $\mathcal{T}$, we can determine the coordinates of t-Feynman points:

$$\begin{aligned}
A_1(t) &:= \mathcal{F}(A_0,B_0,t) = \Big(1-\frac{3\cdot t}{2}, \frac{\sqrt{3}\cdot t}{2}, 0\Big),\\
B_1(t) &:= \mathcal{F}(B_0,C_0,t) = \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2} - \sqrt{3}\cdot t, 0\Big),\\
C_1(t) &:= \mathcal{F}(C_0,D_0,t) = \Big(-\frac{1-t}{2}, -(1-t)\frac{\sqrt{3}}{2}, \sqrt{2}\cdot t\Big),\\
D_1(t) &:= \mathcal{F}(D_0,A_0,t) = (t, 0, \sqrt{2}\cdot(1-t)).
\end{aligned}$$

Given three non-aligned points of $\mathbb{R}^3$, say $A = (a_x,a_y,a_z)$, $B = (b_x,b_y,b_z)$, $C = (c_x,c_y,c_z)$, and using determinants, we define the function $\mathcal{P}(A,B,C)$, that gives the equation of the plane trough A,B,C:

$$\mathcal{P}(A,B,C) = \det\begin{pmatrix} a_x & a_y & a_z & 1\\ b_x & b_y & b_z & 1\\ c_x & c_y & c_z & 1\\ x & y & z & 1\end{pmatrix} = 0.$$

By construction (see Fig. 4), each vertex of $\mathcal{T}_{\mathcal{Q}_1'}(t)$ is the intersection of three planes. Thus, to determine their coordinates we have to solve four systems. The computation gives:

$$\begin{aligned}
A_2(t) &:= \begin{cases}\mathcal{P}(A_0,B_0,C_1(t))\\ \mathcal{P}(B_0,C_0,D_1(t))\\ \mathcal{P}(C_0,D_0,A_1(t))\end{cases}\\
&= \left(\frac{(t^2+t-1)(1-2t)}{2\left(t^2+(1-t)^2\right)}, -\frac{\sqrt{3}\left(t^3+(1-t)^3\right)}{2\left(t^2+(1-t)^2\right)}, \frac{\sqrt{2}\cdot t(1-t)^2}{t^2+(1-t)^2}\right),\\
B_2(t) &:= \begin{cases}\mathcal{P}(B_0,C_0,D_1(t))\\ \mathcal{P}(C_0,D_0,A_1(t))\\ \mathcal{P}(D_0,A_0,B_1(t))\end{cases}\\
&= \left(\frac{t(2-t)(1-2t)}{2\left(t^2+(1-t)^2\right)}, \frac{\sqrt{3}\cdot t^2(1-2t)}{2\left(t^2+(1-t)^2\right)}, \frac{\sqrt{2}\cdot(1-t)^3}{t^2+(1-t)^2}\right),\\
C_2(t) &:= \begin{cases}\mathcal{P}(C_0,D_0,A_1(t))\\ \mathcal{P}(D_0,A_0,B_1(t))\\ \mathcal{P}(A_0,B_0,C_1(t))\end{cases}\\
&= \left(\frac{(1-t)(2-t)(1-2t)}{2\left(t^2+(1-t)^2\right)}, \frac{\sqrt{3}\cdot t(1-2t)(1-t)^2}{2\left(t^2+(1-t)^2\right)}, \frac{\sqrt{2}\cdot t^3}{t^2+(1-t)^2}\right),
\end{aligned} \tag{5}$$

$$(5')\qquad \begin{aligned} D_2(t) &:= \begin{cases} \mathcal{P}(D_0, A_0, B_1(t)) \\ \mathcal{P}(A_0, B_0, C_1(t)) \\ \mathcal{P}(B_0, C_0, D_1(t)) \end{cases} \\ &= \left(\frac{(1+t^2)(2t-1)}{2\,(t^2+(1-t)^2)}, \frac{\sqrt{3}\cdot(1-2t)(1-t)^2}{2\,(t^2+(1-t)^2)}, \frac{\sqrt{2}\cdot t^2(1-t)}{t^2+(1-t)^2} \right). \end{aligned}$$

Finally, we need a function to compute the volume of the tetrahedron $\mathcal{T} = \{A, B, C, D\}$,

$$\operatorname{vol}(A,B,C,D) = \frac{1}{6}\cdot\left|\det\begin{pmatrix} a_x & a_y & a_z & 1 \\ b_x & b_y & b_z & 1 \\ c_x & c_y & c_z & 1 \\ d_x & d_y & d_z & 1 \end{pmatrix}\right|.$$

In particular, for $\mathcal{T} = \{A_0, B_0, C_0, D_0\}$ and $\mathcal{T}_{\mathcal{Q}_1'}(t)) = \{A_2, B_2, C_2, D_2\}$, we obtain:

$$\operatorname{vol}(\mathcal{T}) = \frac{1}{6}\cdot\left|\det\begin{pmatrix} 1 & 0 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & 0 & \sqrt{2} & 1 \end{pmatrix}\right| = \frac{\sqrt{6}}{4}, \quad \text{and}$$

$$\begin{aligned} \operatorname{vol}(\mathcal{T}_{\mathcal{Q}_1'}(t))) &= \frac{1}{6}\cdot\left|\det\begin{pmatrix} \frac{(t^2+t-1)(1-2t)}{2(t^2+(1-t)^2)} & -\frac{\sqrt{3}\left(t^3+(1-t)^3\right)}{2(t^2+(1-t)^2)} & \frac{\sqrt{2}\cdot t(1-t)^2}{t^2+(1-t)^2)} & 1 \\ \frac{t(2-t)(1-2t)}{2(t^2+(1-t)^2)} & \frac{\sqrt{3}\cdot t^2(1-2t)}{2(t^2+(1-t)^2)} & \frac{\sqrt{2}\cdot(1-t)^3}{t^2+(1-t)^2)} & 1 \\ \frac{(1-t)(2-t)(1-2t)}{2(t^2+(1-t)^2)} & \frac{\sqrt{3}\cdot t(1-2t)(1-t)^2}{2(t^2+(1-t)^2)} & \frac{\sqrt{2}\cdot t^3}{t^2+(1-t)^2)} & 1 \\ \frac{(1+t^2)(2t-1)}{2(t^2+(1-t)^2)} & \frac{\sqrt{3}\cdot(1-2t)(1-t)^2}{2(t^2+(1-t)^2)} & \frac{\sqrt{2}\cdot t^2(1-t)}{t^2+(1-t)^2)} & 1 \end{pmatrix}\right| \\ &= \frac{\sqrt{6}}{4}\cdot\left|\frac{(1-2t)^3}{t^2+(1-t)^2}\right|. \end{aligned}$$

Therefore the ratio between the volumes of $\mathcal{T} = (A_0, B_0, C_0, D_0)$ and the t-Feynman's tetrahedron $\mathcal{T}_{\mathcal{Q}_1'}(t) = (A_2(t), B_2(t), C_2(t), D_2(t))$ is

$$(6)\qquad \frac{\operatorname{vol}(A_0, B_0, C_0, D_0)}{\operatorname{vol}(A_2(t), B_2(t), C_2(t), D_2(t))} = \frac{\frac{\sqrt{6}}{4}}{\frac{\sqrt{6}}{4}\cdot\frac{(1-2t)^3}{t^2+(1-t)^2}} = \left|\frac{t^2+(1-t)^2}{(1-2t)^3}\right|.$$

The above argument also runs over for every other choice of ordered skew quadrilaterals $\mathcal{Q}$.

To complete the proof, it is enough to apply Eq. (6):

$$\begin{aligned} \operatorname{vol}(T_{\mathcal{Q}}(1-t)) &= \frac{\sqrt{6}}{4}\cdot\left|\frac{[1-2(1-t)]^3}{(1-t)^2+(1-(1-t))^2}\right| = \left|\frac{[-1+2t)^3}{(1-t)^2+(t))^2}\right| \\ &= \frac{\sqrt{6}}{4}\cdot\left|\frac{(1-2t)^3}{t^2+(1-t)^2}\right| = \operatorname{vol}(T_{\mathcal{Q}}(t)). \quad \blacksquare \end{aligned}$$

The first remarkable consequence of Theorem 3.1 is that the volume of the t-Feynman tetrahedra does not depend on the initial choice of the skew ordered quadrilateral.

Corollary 3.2. *Let $\mathcal{T}$ be a tetrahedron. Then, for every $t < 1$ (positive number), all t-Feynman's tetrahedra of $\mathcal{T}$ have the same volume.*

Remark 3.3. In Remark 2.1, we have observed that $\mathcal{T}_{\mathcal{Q}'_1}(t)$ and $\mathcal{T}_{\mathcal{Q}''_1}(1-t)$ are congruent, therefore they have the same volume. This can be also seen as a consequence of Theorem 3.1.

We conclude this section showing that every t-Feynman's tetrahedron of a regular tetrahedron has four isosceles (non equilateral) faces, in particular it is not regular.

Theorem 3.4. *Let $\mathcal{T} = \{A_0, B_0, C_0, D_0\}$ be a regular tetrahedron and let $t \in (0,1)$. Then, every t-Feynman's tetrahedron of $\mathcal{T}$ has four isosceles faces, in particular it is not regular.*

Proof. We start from a regular tetrahedron $\mathcal{T}$ as defined in (3). Let $Q'_1 = (A_0, B_0, C_0, D_0)$ be an ordered quadrilateral associated with $\mathcal{T}$. We will show that $\mathcal{T}_{\mathcal{Q}'_1}(t)$ has four isosceles faces. Let $A = (a_x, a_y, a_z)$ and $B = (b_x, b_y, b_z)$ be two distinct points in $\mathbb{R}^3$. The length of AB is:

$$\mathcal{D}(A,B) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2}.$$

Using relations (5) and (5′), by a computation we can determine the lengths of the sides of $\mathcal{T}_{\mathcal{Q}'_1}(t)$:

$$\begin{aligned} &\mathcal{D}(\mathcal{A}_2(t)C_2(t)) = \mathcal{D}(\mathcal{B}_2(t)D_2(t)) = \frac{\sqrt{3}\cdot(1-2t))}{\sqrt{t^2+(1-t)^2}} \quad \text{and} \\ &\mathcal{D}(\mathcal{A}_2(t)B_2(t)) = \mathcal{D}(\mathcal{B}_2(t)C_2(t)) = \mathcal{D}(\mathcal{C}_2(t)D_2(t)) = \mathcal{D}(\mathcal{D}_2(t)A_2(t)) \\ &\qquad = \sqrt{t^2-t+1}\cdot\frac{\sqrt{3}\cdot(1-2t))}{\sqrt{t^2+(1-t)^2}} \end{aligned} \tag{7}$$

It turns out that $\mathcal{T}_{\mathcal{Q}'_1}(t)$ has four isosceles faces. Moreover, by the relations (7), $\mathcal{T}_{\mathcal{Q}'_1}(t)$ is regular if and only if $t^2 - t + 1 = 1$. This is impossible as by hypothesis $t \neq 0$ and $t \neq 1$. ∎

4. Applications and remarks

As a special case of Theorem 3.1, we have the corresponding Feynman's theorem for regular tetrahedra:

Corollary 4.1. *Let $\mathcal{T}$ be a (regular) tetrahedron, and let $\mathcal{Q}$ be an associated ordered skew quadrilateral of $\mathcal{T}$. Then*

$$\frac{\mathrm{vol}\,(\mathcal{T})}{\mathrm{vol}\,(\mathcal{T}_{\mathcal{Q}}(1/3))} = \frac{\frac{1}{9}+\frac{4}{9}}{(1-2\cdot\frac{1}{3})^3} = 15.$$

The formula (2) also implies other consequences.

Let $\mathcal{T}$ be a regular tetrahedron, and let $t < 1$ be a positive number. By Theorem 3.1, if t is rational, then the ratio between the volume of $\mathcal{T}$ and the volume of a t-Feynman tetrahedron is likewise rational. In particular, it may be surprising that Eq. (2) gives integer ratios when $t = \frac{1}{3}$ and $t = \frac{1}{4}$. Are there other cases?

As we will see, there are infinite rational parameters such that the ratio of (2) is an integer. To see this, it is helpful to replace the parameter t that we have used in our construction, by another parameter, say u, referred to the middle point of each side. More precisely, if C belongs to a segment AB, and $AC = t \cdot AB$, then there exists a parameter u such that

$$CM = \frac{AM}{u},$$

where M is the middle-point of AB (see Fig. 7).

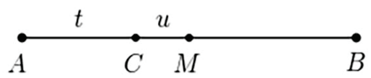

Fig. 7. The relation between the parameters t and u for a point $C \in AB$

Note that the following inequalities hold: $0 < t < 1$ and $u > 1$; moreover, $AM = AC + CM$, thus factoring by AB we have

$$\frac{1}{2} = \frac{t \cdot AB}{AB} + \frac{AM}{uAB} = t + \frac{1}{2u} \implies t = \frac{u-1}{2u}.$$

THEOREM 4.2. *Let u be a positive integer; then for every rational number $t = \dfrac{u-1}{2u}$, the ratio $R(t) = \dfrac{t^2 + (1-t)^2}{2(1-2t)^3}$ is a positive integer.*

Proof.

$$\begin{aligned} R(t) = R\Big(\frac{u-1}{2u}\Big) &= \frac{t^2 + (1-t)^2}{(1-2t)^3} = \frac{2t^2 - 2t + 1}{(1-2t)^3} = \frac{2(\frac{u-1}{2u})^2 - 2(\frac{u-1}{2u}) + 1}{(1 - 2(\frac{u-1}{2u}))^3} \\ &= \frac{\frac{u^2-2u+1}{2u^2} - \frac{u-1}{u} + 1}{\frac{1}{u^3}} = \frac{u^2+1}{2u^2} u^3 = \frac{u(u^2+1)}{2}. \quad \blacksquare \end{aligned} \tag{8}$$

u	1	2	3	4	5	6	7	8	9
$t = \dfrac{u-1}{2u}$	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{5}{12}$	$\frac{3}{7}$	$\frac{7}{15}$	$\frac{4}{9}$
$R\Big(\dfrac{u-1}{2u}\Big) = \dfrac{u(u^2+1)}{2}$	1	5	15	34	65	111	175	260	369

Table 1. Table of ratios between the volume of a tetrahedron and some Feynman's tetrahedra

Remark 4.3. The sequence that comes out from (8), whose initial terms are shown in the last line of the above table, can be found in the Encyclopedia of Integer Sequences (see [12, ref. A006003]), where many of its properties are listed, but not its connection with ratios between volumes of tetrahedra.

We conclude this section by observing that there are also irrational parameters t, such that the ratio in (6) is still an integer.

Example 4.4. Looking at Eq. (6), we consider the following one, imposing the denominator twice the numerator:

$$t^2 + (1-t)^2 = 2(1-2t)^3 \iff f(t) = 16t^3 - 22t^2 + 10t - 1 = 0.$$

Note that $f(0) = -1$ and $f(1/2) > 0$, so that this equation has a real root, say $\bar{t} \in (0, 1/2)$. On the other hand, by Gauss Lemma (see, e.g., [6, p. 41], and [13]) it has no rational roots. This means that there are tetrahedra $\mathcal{T}$ whose volume is twice the volume of its $\bar{t}$-Feynman tetrahedron $\mathcal{T}_{Q_i}(\bar{t})$ where $\bar{t}$ is irrational.

5. Conclusions

The proofs of our results are of the analytical-computational type and make use of affine transformations. On the other hand, just as in the case of the classical Feynman theorem ([3], [10] and reference therein), it is reasonable to think that alternative proofs might exist.

The study of solids and spatial geometry is fundamental both for students' education and from a didactic perspective for several reasons, such as: development of spatial thinking; practical applications and interdisciplinarity; development of abstract thinking and problem-solving skills; relationship with Algebra and other areas of Mathematics (see [1, 5, 7, 8]).

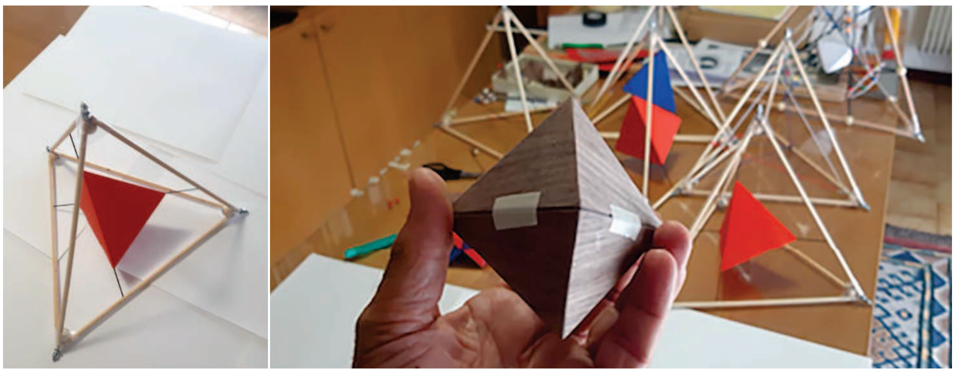

Fig. 8. Some hand-models of derived tetrahedra

The above investigation has been object of laboratory activities for undergraduate students in a regular university course. The combined use of symbolic computational softwares such as Derive or Wolfram-alpha, the use of software such as

GeoGebra (3D modeling), and activities with tangible materials (cardboard solids, 3D printing) make learning more interactive and engaging.

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