THE TEACHING OF MATHEMATICS
2025, Vol. XXVIII, No. 2, pp. 86–99
DOI: 10.57016/TM-HLUM6570

ANGLES AND TRIGONOMETRY

Indrek Zolk

Abstract. In this note, we introduce the sine and cosine functions on (abstract) angles that are defined as equivalence classes of vector pairs. We avoid power series, differential and integral calculus. The number π emerges as the limit of repeated application of half-angle formula (the Viète formula). It is shown that the functions defined coincide with ordinary sine and cosine.

MathEduc Subject Classification: G65, I75
AMS Subject Classification: 97G60, 97I70
Key words and phrases: Angle; trigonometric function; Euclidean dot product.

1. Introduction

A quite typical treatment of trigonometric functions is the following. At school, these functions are first defined in a right triangle (i.e., for angles in $[0, \frac{\pi}{2}]$). After that, they are expanded to the whole real line, perhaps using unit vectors rotating around the unit circle. This treatment is not rigorous (as is school geometry, in general).

In university-level mathematics, however, the students are either told that "basic definitions were already given at school" or these functions are defined rigorously using power series, differential equations (e.g., [1], [3]), or arclength or area of a sector [4], or, as a minimalistic alternative, using functional equations.

This note defines angles *at first*, defines the sine and cosine for angles, and investigates some of their properties in the manner that they eventually coincide with the "ordinary" sine and cosine functions. The derivative, power series, differential equations, the concepts of arc length or area are avoided completely in the further sections. The reasoning will be mostly done in the Euclidean dot product on $\mathbb{R}^2$; however, elements of the exposition can be used for some other dot products.

A number of results in this note are such that their proofs are straightforward verification. For these results, the proofs have been omitted.

Our goal is to find functions $s, c\colon \mathbb{R} \to \mathbb{R}$ satisfying the assumptions of the following theorem.

THEOREM 1. [Main result of [5]] *For any pair of non-constant continuous functions $s, c\colon \mathbb{R} \to \mathbb{R}$ satisfying the two conditions*

(1) $\forall x, y \in \mathbb{R} \quad c(x-y) = c(x)c(y) + s(x)s(y),$

(2) $\lim_{x\to 0+} \dfrac{s(x)}{x} = 1,$

there holds

$$s(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad c(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad x \in \mathbb{R},$$

i.e., s and c coincide with the "usual" sine and cosine, respectively.

2. Non-oriented angles

Let E be any Euclidean space, i.e., a linear space over $\mathbb{R}$ with dot product $\langle \cdot, \cdot \rangle$ that is bilinear, symmetric, and positive for any non-zero vector dot-multiplied by itself. Denote $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$ for all $\boldsymbol{x} \in E$.

Denote $X = E \setminus \{\mathbf{0}\}$. Consider a relation on $X \times X$ as follows:

$$(\boldsymbol{x}, \boldsymbol{u}) \sim (\boldsymbol{y}, \boldsymbol{v}) \quad \Leftrightarrow \quad \frac{\langle \boldsymbol{x}, \boldsymbol{u} \rangle}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|} = \frac{\langle \boldsymbol{y}, \boldsymbol{v} \rangle}{\|\boldsymbol{y}\| \cdot \|\boldsymbol{v}\|}.$$

Proposition 1. *The relation $\sim$ is an equivalence relation.*

In order to avoid brackets, we denote an equivalence class by $[\boldsymbol{x}, \boldsymbol{u}]$ instead of $[(\boldsymbol{x}, \boldsymbol{u})]$.

Proposition 2. [Cauchy-Schwarz inequality] *For any two vectors $\boldsymbol{x}, \boldsymbol{u} \in X$ there holds*

$$-1 \leq \frac{\langle \boldsymbol{x}, \boldsymbol{u} \rangle}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|} \leq 1.$$

Proof. We expand the left hand-side of

$$\left\langle \boldsymbol{x} - \frac{\langle \boldsymbol{x}, \boldsymbol{u} \rangle}{\|\boldsymbol{u}\|} \boldsymbol{u}, \boldsymbol{x} - \frac{\langle \boldsymbol{x}, \boldsymbol{u} \rangle}{\|\boldsymbol{u}\|} \boldsymbol{u} \right\rangle \geq 0. \quad \blacksquare$$

Definition 1. We say that a *non-oriented angle* between two vectors $\boldsymbol{x}, \boldsymbol{u} \in X$ is the equivalence class $[\boldsymbol{x}, \boldsymbol{u}]$. Denote by $N_E = X \times X/\!\sim$ the set of non-oriented angles.

The *cosine* of a non-oriented angle is

$$\mathrm{Cos}[\boldsymbol{x}, \boldsymbol{u}] := \frac{\langle \boldsymbol{x}, \boldsymbol{u} \rangle}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|}.$$

The case $\mathrm{Cos}[\boldsymbol{x}, \boldsymbol{u}] = 0$ is denoted, as usual, by $\boldsymbol{x} \perp \boldsymbol{u}$.

We emphasize that $[\boldsymbol{x}, \boldsymbol{u}] = [\boldsymbol{u}, \boldsymbol{x}]$ since the dot product is commutative.

Next, we verify that if (either) vector is multiplied by positive scalar or if both vectors are transformed orthogonally, the angle remains the same.

Proposition 3. *Let $k > 0$. Then $[k\boldsymbol{x}, \boldsymbol{u}] = [\boldsymbol{x}, \boldsymbol{u}]$.*

Proposition 4. *Let $T\colon E \to E$ be an orthogonal transformation, i.e., a linear transformation that preserves the dot product. Then $[T(\boldsymbol{x}), T(\boldsymbol{u})] = [\boldsymbol{x}, \boldsymbol{u}]$.*

Orthogonal transformations include rotations and reflections, in general, all transformations that preserve angles and vector lengths.

To proceed with the definition of the sine of an angle, we need the oriented angles since we expect the sine to be an odd function. All further discussion will be done for the case $E = \mathbb{R}^2$.

3. Angles

Denote $X = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ and consider a relation on $X \times X$ as follows (denote $\boldsymbol{x} = (x_1, x_2), \boldsymbol{u} = (u_1, u_2), \boldsymbol{y} = (y_1, y_2), \boldsymbol{v} = (v_1, v_2)$):

$$(\boldsymbol{x}, \boldsymbol{u}) \approx (\boldsymbol{y}, \boldsymbol{v}) \quad \Leftrightarrow \quad \begin{cases} \frac{x_1u_1 + x_2u_2}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|} = \frac{y_1v_1 + y_2v_2}{\|\boldsymbol{y}\| \cdot \|\boldsymbol{v}\|}, \\ \frac{x_1u_2 - x_2u_1}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|} = \frac{y_1v_2 - y_2v_1}{\|\boldsymbol{y}\| \cdot \|\boldsymbol{v}\|}. \end{cases}$$

Proposition 5. *The relation $\approx$ is an equivalence relation.*

Definition 2. We say that the *angle* between two vectors $\boldsymbol{x}, \boldsymbol{u} \in X$ is the equivalence class $[\![\boldsymbol{x}, \boldsymbol{u}]\!] := [\![(\boldsymbol{x}, \boldsymbol{u})]\!]$. Denote by $A = X \times X/\approx$ the set of angles.

The *sine* and the *cosine* of the angle $[\![\boldsymbol{x}, \boldsymbol{u}]\!]$ are

$$\mathrm{Sin}[\![\boldsymbol{x}, \boldsymbol{u}]\!] := \frac{x_1u_2 - x_2u_1}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|}, \quad \mathrm{Cos}[\![\boldsymbol{x}, \boldsymbol{u}]\!] := \frac{x_1u_1 + x_2u_2}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|},$$

respectively.

We see that if $\tilde{\boldsymbol{u}} = (u_2, -u_1)$ then $\mathrm{Sin}[\![\boldsymbol{x}, \boldsymbol{u}]\!] = \mathrm{Cos}[\![\boldsymbol{x}, \tilde{\boldsymbol{u}}]\!]$. As the correctness of the definitions of sine and cosine follows from the fact that $\approx$ is equivalence relation, due to Prop. 2, we have defined the functions $\mathrm{Sin}, \mathrm{Cos}\colon A \to [-1, 1]$.

Note that again angle is invariant of the length of vectors. For orthogonal transformations, we must be more careful since, in general, $(\boldsymbol{x}, \boldsymbol{u}) \not\approx (\boldsymbol{u}, \boldsymbol{x})$. We have the following.

Proposition 6. *Let $k > 0$. Then $[\![k\boldsymbol{x}, \boldsymbol{u}]\!] = [\![\boldsymbol{x}, \boldsymbol{u}]\!]$.*

Proposition 7. *Let a, b be real numbers such that $a^2 + b^2 = 1$. Let*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} b & -a \\ a & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} b & -a \\ a & b \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then $(\boldsymbol{y}, \boldsymbol{v}) \approx (\boldsymbol{x}, \boldsymbol{u})$, and $\|\boldsymbol{y}\| = \|\boldsymbol{x}\|$, $\|\boldsymbol{v}\| = \|\boldsymbol{u}\|$.

Proof. We have

$$y_1 = bx_1 - ax_2, \quad y_2 = ax_1 + bx_2, \quad v_1 = bu_1 - au_2, \quad v_2 = au_1 + bu_2,$$

hence

$$y_1^2 + y_2^2 = (bx_1 - ax_2)^2 + (ax_1 + bx_2)^2 = (a^2 + b^2)(x_1^2 + x_2^2) = x_1^2 + x_2^2,$$

analogously $v_1^2 + v_2^2 = u_1^2 + u_2^2$, and

$$\begin{aligned}\frac{y_1v_1 + y_2v_2}{\|\boldsymbol{y}\| \cdot \|\boldsymbol{v}\|} &= \frac{(bx_1 - ax_2)(bu_1 - au_2) + (ax_1 + bx_2)(au_1 + bu_2)}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|} \\ &= \frac{(a^2 + b^2)(x_1u_1 + x_2u_2)}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|} = \frac{x_1u_1 + x_2u_2}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|},\end{aligned}$$

analogously $\frac{y_1v_2 - y_2v_1}{\|\boldsymbol{y}\| \cdot \|\boldsymbol{v}\|} = \frac{x_1u_2 - x_2u_1}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{u}\|}$. ■

The matrices of the form $\begin{pmatrix} b & -a \\ a & b \end{pmatrix}$ as in Prop. 7 will be called *rotation matrices.* The component vectors $\boldsymbol{x}, \boldsymbol{u} \in X$ of the angle $\alpha = [\![\boldsymbol{x}, \boldsymbol{u}]\!]$ will occasionally be called the *legs* of angle α.

Note that we can always choose the representative of the form $((1, 0), \boldsymbol{u})$ (in general, we can choose the first leg to be any non-zero vector). Indeed, for any $(\boldsymbol{x}, \boldsymbol{u}) \in X \times X$, $x_1^2 + x_2^2 = 1$, we can find the respective rotation matrix: (a, b) is the unique solution of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ a & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 & x_1 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Assume from now on that all representatives of equivalence classes are pairs of length 1 vectors. This can be achieved as in view of Prop. 6, scaling vectors leaves the angle unchanged.

Proposition 8. *We have* $\operatorname{Sin}[\![(1, 0), (u_1, u_2)]\!] = u_2$, $\operatorname{Cos}[\![(1, 0), (u_1, u_2)]\!] = u_1$.

4. Sum of angles

The following definition has been inspired by adding angles visually: the second angle should be rotated in the manner such that its first leg coincides with the second leg of the first angle. The sum is the angle between the first leg of the first angle and the second leg of the second angle. (Bear in mind that all legs are assumed to be unit vectors.)

Definition 3. Let $[\![\boldsymbol{x}, \boldsymbol{u}]\!], [\![\boldsymbol{u}, \boldsymbol{v}]\!] \in A$. We define the *sum* of angles $[\![\boldsymbol{x}, \boldsymbol{u}]\!]$ and $[\![\boldsymbol{u}, \boldsymbol{v}]\!]$ by

$$[\![\boldsymbol{x}, \boldsymbol{u}]\!] + [\![\boldsymbol{u}, \boldsymbol{v}]\!] = [\![\boldsymbol{x}, \boldsymbol{v}]\!].$$

Note that $[\![\boldsymbol{x}, \boldsymbol{u}]\!] + [\![\boldsymbol{u}, \boldsymbol{x}]\!] = [\![(1, 0), (1, 0)]\!]$.

In order to simplify calculations, we derive the expression of the sum of angles expressed using arbitrary length one vector pairs.

Theorem 2. For any two angles $[\![\boldsymbol{x}, \boldsymbol{u}]\!], [\![\boldsymbol{y}, \boldsymbol{v}]\!]$ there holds

$$\begin{aligned}&[\![\boldsymbol{x}, \boldsymbol{u}]\!] + [\![\boldsymbol{y}, \boldsymbol{v}]\!] \\ &= [\![\boldsymbol{x}, (u_2v_1y_2 + u_2v_1y_2 - u_2v_2y_1 + u_1v_1y_1, u_2v_2y_2 - u_1v_1y_2 + u_1v_2y_1 + u_2v_1y_1)]\!].\end{aligned}$$

In particular,

$$\llbracket(1,0),\boldsymbol{u}\rrbracket+\llbracket(1,0),\boldsymbol{v}\rrbracket=\llbracket(1,0),(-u_2v_2+u_1v_1,u_1v_2+u_2v_1)\rrbracket.$$

Proof. We use the suitable rotation matrix for the representative of the second angle and apply the definition of the sum. ■

PROPOSITION 9. *For any angle* $\llbracket(1,0),\boldsymbol{u}\rrbracket$*, we have*

$$\llbracket(1,0),\boldsymbol{u}\rrbracket+\llbracket(1,0),(1,0)\llbracket=\llbracket(1,0),\boldsymbol{u}\rrbracket,$$
$$\llbracket(1,0),\boldsymbol{u}\rrbracket+\llbracket(1,0),(u_1,-u_2)\rrbracket=\llbracket(1,0),(1,0)\rrbracket.$$

DEFINITION 4. We denote $0=\llbracket(1,0),(1,0)\rrbracket$ and call it *zero angle*.

For any angle $\alpha=\llbracket(1,0),(u_1,u_2)\rrbracket$, we call the angle $\llbracket(1,0),(u_1,-u_2)\rrbracket$ the *conjugate angle* of α and denote it by $-\alpha$.

For angles $\alpha,\beta\in A$ denote $\alpha-\beta=\alpha+(-\beta)$.

THEOREM 3. $(A,+)$ *is an Abelian group.*

Proof. It is perhaps easiest to prove associativity by

$$\begin{aligned}(\llbracket\boldsymbol{x},\boldsymbol{u}\rrbracket+\llbracket\boldsymbol{u},\boldsymbol{v}\rrbracket)+\llbracket\boldsymbol{v},\boldsymbol{w}\rrbracket&=\llbracket\boldsymbol{x},\boldsymbol{v}\rrbracket+\llbracket\boldsymbol{v},\boldsymbol{w}\rrbracket=\llbracket\boldsymbol{x},\boldsymbol{w}\rrbracket=\\&=\llbracket\boldsymbol{x},\boldsymbol{u}\rrbracket+\llbracket\boldsymbol{u},\boldsymbol{w}\rrbracket=\llbracket\boldsymbol{x},\boldsymbol{u}\rrbracket+(\llbracket\boldsymbol{u},\boldsymbol{v}\rrbracket+\llbracket\boldsymbol{v},\boldsymbol{w}\rrbracket).\end{aligned}$$

Commutativity is obvious, the zero element is the zero angle and, for any angle, its inverse element is its conjugate angle. ■

For any $n\in\mathbb{Z}$ and $\alpha\in A$, we denote

$$n\alpha=\begin{cases}\underbrace{\alpha+\ldots+\alpha}_{n\text{ addends}}, & \text{if } n>0,\\ 0, & \text{if } n=0,\\ -(-n)\alpha, & \text{if } n<0.\end{cases}$$

PROPOSITION 10. *For all* $\alpha,\beta\in A$*, there hold*

$$\begin{aligned}\operatorname{Sin}(-\alpha)&=-\operatorname{Sin}\alpha,\\ \operatorname{Cos}(-\alpha)&=\operatorname{Cos}\alpha,\\ \operatorname{Sin}(\alpha\pm\beta)&=\operatorname{Sin}\alpha\operatorname{Cos}\beta\pm\operatorname{Sin}\beta\operatorname{Cos}\alpha,\\ \operatorname{Cos}(\alpha\pm\beta)&=\operatorname{Cos}\alpha\operatorname{Cos}\beta\mp\operatorname{Sin}\alpha\operatorname{Sin}\beta,\\ \operatorname{Sin}2\alpha&=2\operatorname{Sin}\alpha\operatorname{Cos}\alpha,\\ \operatorname{Cos}2\alpha&=(\operatorname{Cos}\alpha)^2-(\operatorname{Sin}\alpha)^2.\end{aligned}$$

Proof. As an example, we prove that $\operatorname{Cos}(\alpha-\beta)=\operatorname{Cos}\alpha\operatorname{Cos}\beta+\operatorname{Sin}\alpha\operatorname{Sin}\beta$.

Let $\alpha=\llbracket(1,0),\boldsymbol{u}\rrbracket$ and $\beta=\llbracket(1,0),\boldsymbol{v}\rrbracket$. Now $-\beta=\llbracket(1,0),(v_1,-v_2)\rrbracket$. Due to Theorem 2,

$$\alpha-\beta=\alpha+(-\beta)=\llbracket(1,0),(u_2v_2+u_1v_1,-u_1v_2+u_2v_1)\rrbracket.$$

Using Prop. 8, we conclude that

$$\operatorname{Cos}(\alpha-\beta)=u_2v_2+u_1v_1=\operatorname{Sin}\alpha\operatorname{Sin}\beta+\operatorname{Cos}\alpha\operatorname{Cos}\beta. \quad\blacksquare$$

Call $\urcorner=\llbracket(1,0),(0,1)\rrbracket$ the *right angle*. We have $\operatorname{Cos}\urcorner=0$ and $\operatorname{Sin}\urcorner=1$.
Call $\curvearrowleft=\llbracket(1,0),(-1,0)\rrbracket$ the *straight angle*. We have $\operatorname{Cos}\curvearrowleft=-1$ and $\operatorname{Sin}\curvearrowleft=0$.

PROPOSITION 11. *For any $\alpha\in A$, all angles $\beta\in A$ that satisfy $2\beta=\alpha$, are of the form*

$$\left\llbracket(1,0),\pm\left(\sqrt{\frac{1+\operatorname{Cos}\alpha}{2}},(\operatorname{sgn}\operatorname{Sin}\alpha)\sqrt{\frac{1-\operatorname{Cos}\alpha}{2}}\right)\right\rrbracket,$$

i.e., of the form

$$\left\llbracket(1,0),\left(\sqrt{\frac{1+\operatorname{Cos}\alpha}{2}},(\operatorname{sgn}\operatorname{Sin}\alpha)\sqrt{\frac{1-\operatorname{Cos}\alpha}{2}}\right)\right\rrbracket+k\cdot\curvearrowleft,\qquad k\in\{0,1\}.$$

Proof. In order to find all angles β for which $\beta+\beta=\alpha$, we take $\beta=\llbracket(1,0),\boldsymbol{u}\rrbracket$, find $2\beta=\llbracket(1,0),(u_1^2-u_2^2,2u_1u_2)\rrbracket$, and solve algebraically the system of equations

$$(u_1^2-u_2^2,2u_1u_2)=(\operatorname{Cos}\alpha,\operatorname{Sin}\alpha),\qquad u_1^2+u_2^2=1.\quad\blacksquare$$

Define

$$A_0=\{\alpha\in A\ :\ \operatorname{Sin}\alpha\in[0,1],\ \operatorname{Cos}\alpha\in[0,1]\}.$$

Due to Prop. 12, the sine and cosine functions are periodic with period $4\urcorner$ and the values in the quadrant A_0 determine uniquely all their values.

PROPOSITION 12. *For all $\alpha\in A$, there hold*

$$\begin{aligned}\operatorname{Sin}(\urcorner-\alpha)&=\operatorname{Cos}\alpha,\\ \operatorname{Cos}(\urcorner-\alpha)&=\operatorname{Sin}\alpha,\\ \operatorname{Sin}(2\cdot\urcorner-\alpha)&=\operatorname{Sin}\alpha,\\ \operatorname{Cos}(2\cdot\urcorner-\alpha)&=-\operatorname{Cos}\alpha,\end{aligned}\qquad \begin{aligned}\operatorname{Sin}(2\cdot\urcorner+\alpha)&=-\operatorname{Sin}\alpha,\\ \operatorname{Cos}(2\cdot\urcorner+\alpha)&=-\operatorname{Cos}\alpha,\\ \operatorname{Sin}(4\cdot\urcorner-\alpha)&=-\operatorname{Sin}\alpha,\end{aligned}\qquad \begin{aligned}\operatorname{Cos}(4\cdot\urcorner-\alpha)&=\operatorname{Cos}\alpha,\\ \operatorname{Sin}(4\cdot\urcorner+\alpha)&=\operatorname{Sin}\alpha,\\ \operatorname{Cos}(4\cdot\urcorner+\alpha)&=\operatorname{Cos}\alpha.\end{aligned}$$

5. Dyadic values and π

In this section, we shall determine many values of sine and cosine in the first quadrant. Note that due to Prop. 8, for $\alpha = [\![(1,0), \boldsymbol{u}]\!]$ where $u_1, u_2 \in (0,1)$, we have $\operatorname{Sin}\alpha = u_2 > 0$ and $\operatorname{Cos}\alpha = u_1 > 0$.

Definition 5. For any $\alpha \in A_0$, denote $\frac{\alpha}{2} = \left[\!\!\left[(1,0), \left(\sqrt{\frac{1+\operatorname{Cos}\alpha}{2}}, \sqrt{\frac{1-\operatorname{Cos}\alpha}{2}}\right)\right]\!\!\right]$. In particular, note that $\frac{\daleth}{2} = \left[\!\!\left[(1,0), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right]\!\!\right]$.

Proposition 13. *For all $\alpha, \beta \in A_0$, there hold*

$$\operatorname{Sin}\alpha \pm \operatorname{Sin}\beta = 2\operatorname{Sin}\frac{\alpha \pm \beta}{2} \cdot \operatorname{Cos}\frac{\alpha \mp \beta}{2},$$
$$\operatorname{Cos}\alpha + \operatorname{Cos}\beta = 2\operatorname{Cos}\frac{\alpha + \beta}{2} \cdot \operatorname{Cos}\frac{\alpha - \beta}{2},$$
$$\operatorname{Cos}\alpha - \operatorname{Cos}\beta = -2\operatorname{Sin}\frac{\alpha + \beta}{2} \cdot \operatorname{Sin}\frac{\alpha - \beta}{2},$$

whenever $\frac{\alpha \pm \beta}{2} \in A_0$.

Def. 5 (see also Prop. 11) allows to calculate the values of sine and cosine for any dyadic multiple of $\daleth$, i.e., for any angle from the set $D \cdot \daleth$ where

$$D = \left\{\frac{m}{2^k} \ : \ k \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z} \cap [0, 2^k]\right\}.$$

Indeed, the steps are the following: halving the right angle k times and then adding m copies of the result.

It is well known that $\overline{D} = [0,1]$, i.e., the set D is dense in $[0,1]$.

Proposition 14. *For every $k \in \mathbb{N}$, we have*

$$\operatorname{Sin}\left(\frac{1}{2^k} \cdot \daleth\right) = \frac{1}{2} \cdot \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{k \text{ radicals}},$$
$$\operatorname{Cos}\left(\frac{1}{2^k} \cdot \daleth\right) = \frac{1}{2} \cdot \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{k \text{ radicals}}.$$

Moreover,

(1) *the sequences* $\left(\operatorname{Sin}\left(\frac{1}{2^k} \cdot \daleth\right)\right)_k$ *and* $\left(\frac{\operatorname{Sin}\left(\frac{1}{2^k} \cdot \daleth\right)}{\operatorname{Cos}\left(\frac{1}{2^k} \cdot \daleth\right)}\right)_k$ *are decreasing and converge to* 0*;*

(2) *the sequence* $\left(2^k \operatorname{Sin}\left(\frac{1}{2^k} \cdot \daleth\right)\right)_k$ *is increasing, the sequence* $\left(2^k \cdot \frac{\operatorname{Sin}\left(\frac{1}{2^k} \cdot \daleth\right)}{\operatorname{Cos}\left(\frac{1}{2^k} \cdot \daleth\right)}\right)_k$ *is decreasing and these two sequences converge to the same limit* $\Pi > 0$.

Proof. The closed formulae can be proved by induction, relying on Prop. 11.

For the "moreover" part (1), denote $a_k = 2\,\mathrm{Sin}\left(\frac{1}{2^k}\cdot \Lsh\right)$ and $b_k = \frac{2\,\mathrm{Sin}\left(\frac{1}{2^k}\cdot\Lsh\right)}{\mathrm{Cos}\left(\frac{1}{2^k}\cdot\Lsh\right)}$, and note that the following recursions hold:

$$\begin{aligned} a_1 &= \sqrt{2}, & a_{k+1} &= \sqrt{2-\sqrt{4-a_k^2}}, \\ b_1 &= 2, & b_{k+1} &= \frac{2b_k}{2+\sqrt{4+b_k^2}}. \end{aligned}$$

Now, $a_2 = \sqrt{2-\sqrt{2}} < \sqrt{2} = a_1$, and since the function $f(x) = \sqrt{2-\sqrt{4-x^2}}$ is increasing on $[0,2]$, we can verify by induction that (a_k) is decreasing. Similarly, $b_2 = \frac{4}{2+2\sqrt{2}} < 2 = b_1$, and $g(x) = \frac{2x}{2+\sqrt{4+x^2}} = 2\left(\frac{2}{x}+\sqrt{1+\frac{4}{x^2}}\right)^{-1}$ is increasing on $[0,2]$, hence (b_n) is decreasing. The limit value can be obtained by taking limits in the recursions.

For part (2), note that $2f(x) > x$ and $2g(x) < x$ for all $x \in (0,2)$, hence the sequences $(2^k a_k)$ and $(2^k b_k)$ are strictly increasing and decreasing, respectively. By induction, one can show that

$$a_k^2 = \frac{4b_k^2}{4+b_k^2}, \quad k \in \mathbb{N}.$$

As $(2^k b_k)$ is convergent (it is decreasing and bounded by 0 from below), we have

$$\lim_k 4^k a_k^2 = \lim_k \frac{4\cdot 4^k b_k^2}{4+b_k^2} = \lim_k 4^k b_k^2. \quad \blacksquare$$

Remark 1. Prop. 14 is essentially IMC 2001 Second day Problem 2 [2].

Remark 2. As the sequences $(2^k a_k)$ and $(2^k b_k)$ are strictly increasing and decreasing, respectively, we have $\Pi > 2a_1 = 2\sqrt{2}$ and $\Pi < 2b_1 = 4$.

Remark 3. The *Viète formula*

$$\frac{2}{\pi} = \lim_k \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot\ldots\cdot\frac{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{k\text{ radicals}}}{2}$$

dates back to already 1593 [6]. Moreover,

$$\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot\ldots\cdot\frac{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{k\text{ radicals}}}{2}$$

$$
\begin{aligned}
&= \left(\operatorname{Cos}\left(\frac{1}{2}\cdot \Lsh\right)\right)\cdot\left(\operatorname{Cos}\left(\frac{1}{2^2}\cdot \Lsh\right)\right)\cdot\left(\operatorname{Cos}\left(\frac{1}{2^3}\cdot \Lsh\right)\right)\cdot\ldots\cdot\left(\operatorname{Cos}\left(\frac{1}{2^k}\cdot \Lsh\right)\right)\\
&= \left(\operatorname{Cos}\left(\frac{1}{2}\cdot \Lsh\right)\right)\cdot\left(\operatorname{Cos}\left(\frac{1}{2^2}\cdot \Lsh\right)\right)\cdot\left(\operatorname{Cos}\left(\frac{1}{2^3}\cdot \Lsh\right)\right)\cdot\ldots\cdot\\
&\quad\left(\operatorname{Cos}\left(\frac{1}{2^k}\cdot \Lsh\right)\right)\cdot\frac{2^k \operatorname{Sin}\left(\frac{1}{2^k}\cdot \Lsh\right)}{2^k \operatorname{Sin}\left(\frac{1}{2^k}\cdot \Lsh\right)}\\
&= \frac{\operatorname{Sin}(\Lsh)}{2^k \operatorname{Sin}\left(\frac{1}{2^k}\cdot \Lsh\right)} = \frac{1}{2^k \operatorname{Sin}\left(\frac{1}{2^k}\cdot \Lsh\right)} = \frac{2}{2^k a_k} \to \frac{2}{\Pi},\ \text{as } k\to\infty.
\end{aligned}
$$

Due to Remark 3, we shall denote $\pi = \Pi$ further on.

Remark 4. In order to find all angles α for which $\alpha+\alpha+\alpha = \Lsh$, we take $\alpha = [\![(1,0), \boldsymbol{u}]\!]$, find $3\alpha = [\![(1,0),(4u_1^3 - 3u_1, 3u_2 - 4u_2^3)]\!]$, and solve algebraically the system of equations

$$(4u_1^3 - 3u_1, 3u_2 - 4u_2^3) = (0,1), \qquad u_1^2 + u_2^2 = 1.$$

We obtain that $(u_1,u_2) \in \{(0,-1),(\pm\frac{\sqrt{3}}{2},\frac{1}{2})\}$. The only such α that belongs to A_0 is $[\![(1,0),(\frac{\sqrt{3}}{2},\frac{1}{2})]\!]$.

6. Sine and cosine on the real line

For a real number a and a positive real number b, we denote by $a \bmod b$ the least non-negative number c such that $b \mid a-c$, i.e., such that there exists an integer x such that $bx = a-c$. (Such a number c always exists since the set $\{x\in\mathbb{Z} \,:\, a - bx \ge 0\}$ is non-empty: we have $a - b\cdot\left\lfloor -|a|\cdot\left\lceil\frac{1}{b}\right\rceil\right\rfloor \ge 0$.)

Consider $(\mathbb{R},+)$ as a group with respect to ordinary addition.

Proposition 15. *For any group homomorphism $\varphi\colon \mathbb{R}\to A$ (i.e., a function that preserves group operation) such that $\varphi(\frac{\pi}{2}) = \Lsh$, its behaviour is fully described by $\varphi|_{[0,\pi]}$. The same applies to the corresponding functions* $\operatorname{Sin}\circ\varphi, \operatorname{Cos}\circ\varphi\colon \mathbb{R}\to\mathbb{R}$.

Proof. Straightforward verification shows that

$$
\begin{aligned}
\varphi(\pi) &= \varphi(\tfrac{\pi}{2}+\tfrac{\pi}{2}) = \varphi(\tfrac{\pi}{2}) + \varphi(\tfrac{\pi}{2}) = \Lsh + \Lsh = \curvearrowleft,\\
\varphi(2\pi) &= 2\varphi(\pi) = \curvearrowleft + \curvearrowleft = 0,
\end{aligned}
$$

hence, for any $x\in\mathbb{R}$, firstly we have $\varphi(x) = \varphi(x \bmod 2\pi)$, and now, assuming that $x\in[0,2\pi)$, we have

$$\varphi(x) = \varphi\left(x \bmod \tfrac{\pi}{2} + \left\lfloor\frac{x}{\frac{\pi}{2}}\right\rfloor\cdot\tfrac{\pi}{2}\right) = \varphi(x \bmod \tfrac{\pi}{2}) + \left\lfloor\frac{x}{\frac{\pi}{2}}\right\rfloor\cdot\Lsh$$

where $\left\lfloor\frac{x}{\frac{\pi}{2}}\right\rfloor = 1,2,3$ if $x\in[\frac{\pi}{2},\pi)$, $x\in[\pi,\frac{3\pi}{2})$, or $x\in[\frac{3\pi}{2},2\pi)$, respectively.

The precise formulae for $\operatorname{Sin}\varphi(x)$ and $\operatorname{Cos}\varphi(x)$ (for arguments $x\in(\frac{\pi}{2},2\pi)$) follow by Prop. 12. ∎

Our goal is to verify that (at least one) such φ exists and that $\mathrm{Sin}\circ\varphi$ and $\mathrm{Cos}\circ\varphi$ satisfy the conditions of Theorem 1.

Proposition 16. *Let $\varphi\colon \mathbb{R} \to A$ be any group homomorphism such that $\varphi(\frac{\pi}{2}) = \urcorner$. For any dyadic multiple of $\frac{\pi}{2}$, $\frac{m}{2^k}\cdot\frac{\pi}{2} \in D\cdot\frac{\pi}{2}$, it holds*

$$\varphi\left(\frac{m}{2^k}\cdot\frac{\pi}{2}\right) = \frac{m}{2^k}\cdot \urcorner\,.$$

Proof. Note that for any $x \in [0, \frac{\pi}{2}]$, we have $\varphi(x) = \varphi(\frac{x}{2}+\frac{x}{2}) = 2\varphi(\frac{x}{2})$, hence inductively $\varphi(\frac{1}{2^k}\cdot\frac{\pi}{2}) = \frac{1}{2^k}\cdot \urcorner$. ■

Theorem 4. *Let $\varphi\colon \mathbb{R} \to A$ be any group homomorphism such that $\varphi(\frac{\pi}{2}) = \urcorner$. For any $x \in (D\cdot\frac{\pi}{2}) \setminus \{0\}$, there holds*

$$0 < x(\mathrm{Cos}\,\varphi(x)) < \mathrm{Sin}\,\varphi(x) < x.$$

Proof. The inequality $0 < \mathrm{Cos}\,\varphi(x)$ is trivial, as $\varphi[(D\cdot\frac{\pi}{2})\setminus\{0\}] \subseteq A_0 \setminus \{0\}$.

The other two inequalities will be proven by induction. The base case is with numerator 1. Assume $x = \frac{1}{2^k}\frac{\pi}{2}$, then $\varphi(x) = \frac{1}{2^k}\cdot \urcorner$. Hence we have to prove that

$$2\cdot 2^k\,\mathrm{Sin}\left(\frac{1}{2^k}\cdot \urcorner\right) < \pi, \quad 2\cdot 2^k\,\frac{\mathrm{Sin}\left(\frac{1}{2^k}\cdot \urcorner\right)}{\mathrm{Cos}\left(\frac{1}{2^k}\cdot \urcorner\right)} > \pi,$$

that is a direct consequence of Prop. 14.

Now, if x and y are of the form $\frac{m}{2^k}\frac{\pi}{2}$ for which the desired inequalities already hold then

$$\begin{aligned}\mathrm{Sin}\,\varphi(x+y) &= \mathrm{Sin}(\varphi(x)+\varphi(y)) = (\mathrm{Sin}\,\varphi(x))(\mathrm{Cos}\,\varphi(y)) + (\mathrm{Cos}\,\varphi(x))(\mathrm{Sin}\,\varphi(y))\\ &\le \mathrm{Sin}\,\varphi(x) + \mathrm{Sin}\,\varphi(y) < x+y,\\ \frac{\mathrm{Sin}\,\varphi(x+y)}{\mathrm{Cos}\,\varphi(x+y)} &= \frac{\mathrm{Sin}(\varphi(x)+\varphi(y))}{\mathrm{Cos}\,\varphi(x)+\varphi(y)} = \frac{\frac{\mathrm{Sin}\,\varphi(x)}{\mathrm{Cos}\,\varphi(x)} + \frac{\mathrm{Sin}\,\varphi(y)}{\mathrm{Cos}\,\varphi(y)}}{1 - \frac{\mathrm{Sin}\,\varphi(x)}{\mathrm{Cos}\,\varphi(x)}\cdot\frac{\mathrm{Sin}\,\varphi(y)}{\mathrm{Cos}\,\varphi(y)}}\\ &\ge \frac{\mathrm{Sin}\,\varphi(x)}{\mathrm{Cos}\,\varphi(x)} + \frac{\mathrm{Sin}\,\varphi(y)}{\mathrm{Cos}\,\varphi(y)} > x+y. \quad ■\end{aligned}$$

The next proposition implies that the sine and cosine are monotone and continuous on dyadic multiples of right angle.

Proposition 17. *Let $\varphi\colon \mathbb{R} \to A$ be any group homomorphism such that $\varphi(\frac{\pi}{2}) = \urcorner$. For $x, y \in D\cdot\frac{\pi}{2}$, there holds:*

$$x < y \quad\Rightarrow\quad \begin{cases}\mathrm{Cos}\,\varphi(x) > \mathrm{Cos}\,\varphi(y),\\ \mathrm{Sin}\,\varphi(x) < \mathrm{Sin}\,\varphi(y),\end{cases}$$

and

$$|\operatorname{Cos}\varphi(x)-\operatorname{Cos}\varphi(y)|\le|x-y|,\qquad|\operatorname{Sin}\varphi(x)-\operatorname{Sin}\varphi(y)|\le|x-y|.$$

Proof. Note that if $t,t'\in D$ (assume $t>t'$) then

$$\frac{\varphi(t\cdot\frac{\pi}{2})\pm\varphi(t'\cdot\frac{\pi}{2})}{2}=\frac{t\cdot\Lsh\pm t'\cdot\Lsh}{2}=\frac{(t\pm t')\cdot\Lsh}{2}=\frac{t\pm t'}{2}\cdot\Lsh=\varphi\left(\frac{t\pm t'}{2}\cdot\frac{\pi}{2}\right),$$

since $\frac{t-t'}{2}\in D$.

Now, if $y>x$ (hence $\frac{y\pm x}{2}\in D\cdot\frac{\pi}{2}$) then due to Prop. 13 and Theorem 4, we have

$$\operatorname{Sin}\varphi(y)-\operatorname{Sin}\varphi(x)=2\left(\operatorname{Cos}\frac{\varphi(y)+\varphi(x)}{2}\right)\cdot\left(\operatorname{Sin}\frac{\varphi(y)-\varphi(x)}{2}\right)>0,$$
$$\operatorname{Cos}\varphi(y)-\operatorname{Cos}\varphi(x)=-2\left(\operatorname{Sin}\frac{\varphi(y)+\varphi(x)}{2}\right)\cdot\left(\operatorname{Sin}\frac{\varphi(y)-\varphi(x)}{2}\right)<0.$$

Therefore

$$\begin{aligned}|\operatorname{Cos}\varphi(x)-\operatorname{Cos}\varphi(y)|&=2\left|\operatorname{Sin}\frac{\varphi(x)+\varphi(y)}{2}\right|\cdot\left|\operatorname{Sin}\frac{\varphi(x)-\varphi(y)}{2}\right|\\&\le 2\cdot\frac{|x-y|}{2}=|x-y|,\end{aligned}$$

analogously for the sine. ■

The following proposition is a direct consequence of Prop. 17.

PROPOSITION 18. *Let $\varphi\colon\mathbb{R}\to A$ be any group homomorphism such that $\varphi(\frac{\pi}{2})=\Lsh$. Let (x_n) and (x'_n) be sequences whose elements belong to $D\cdot\frac{\pi}{2}$. Let $\lim_n(x_n-x'_n)=0$. Then*

$$\lim_n(\operatorname{Cos}\varphi(x_n)-\operatorname{Cos}\varphi(x'_n))=0,\qquad\lim_n(\operatorname{Sin}\varphi(x_n)-\operatorname{Sin}\varphi(x'_n))=0.$$

THEOREM 5. *Let $\varphi\colon\mathbb{R}\to A$ be any group homomorphism such that $\varphi(\frac{\pi}{2})=\Lsh$. The functions $(D\cdot\frac{\pi}{2})\ni x\mapsto\operatorname{Sin}\varphi(x)\in[0,1]$ and $(D\cdot\frac{\pi}{2})\ni x\mapsto\operatorname{Cos}\varphi(x)\in[0,1]$ have uniformly continuous extensions to $[0,\frac{\pi}{2}]$.*

First part of proof of Thm. 5. Prop. 18 yields that for $t\in[0,1]$, the limits

$$\lim_n\operatorname{Cos}\varphi(t_n\cdot\tfrac{\pi}{2}),\quad\lim_n\operatorname{Sin}\varphi(t_n\cdot\tfrac{\pi}{2}),\qquad t_n\in D,\quad t_n\to t. \tag{1}$$

exist and are independent on the choice of the sequence (t_n).

Indeed, since the sequence (t_n) is convergent, it is Cauchy; now an argument similar to that of Prop. 18 yields that the sequences $(\operatorname{Cos}\varphi(t_n\cdot\frac{\pi}{2}))$ and $(\operatorname{Sin}\varphi(t_n\cdot\frac{\pi}{2}))$ are also Cauchy, hence convergent.

Prop. 18 also yields that the limits (1) do not depend on the choice of the sequence.

The proof of (uniform) continuity will be postponed until we denote the extensions.

DEFINITION 6. Define a mapping $\varphi\colon \mathbb{R} \to A$ as follows:

- for all $t \in D$, we define $\varphi(t \cdot \frac{\pi}{2}) = t \cdot \urcorner$,
- for all $x \in [0, \frac{\pi}{2}]$ we define $\varphi(x) = (\lim_n \operatorname{Sin} \varphi(x_n), \lim_n \operatorname{Cos} \varphi(x_n))$ where (x_n) is any sequence in $D \cdot \frac{\pi}{2}$ and converging to x,
- for all $x \in \mathbb{R}$, we define $\varphi(x) = \varphi(x \mod (2\pi))$ and if $x \in [0, 2\pi)$ then $\varphi(x) = \varphi(x \mod \frac{\pi}{2}) + \left\lfloor \frac{x}{\frac{\pi}{2}} \right\rfloor \cdot \urcorner$.

Denote

$$s(x) = \operatorname{Sin} \varphi(x), \quad c(x) = \operatorname{Cos} \varphi(x), \qquad x \in \mathbb{R}.$$

PROPOSITION 19. *The mapping φ defined as in Def. 6 is a group homomorphism.*

Continuation of proof of Thm. 5. Fix $\varepsilon > 0$. We prove that for all $x, y \in \mathbb{R}$,

$$|x - y| < \frac{\varepsilon}{2} \quad \Rightarrow \quad \begin{cases} |s(x) - s(y)| < \varepsilon, \\ |c(x) - c(y)| < \varepsilon. \end{cases}$$

It suffices to consider $[0, \frac{\pi}{2}]$ only. Take sequences $(x_n), (y_n)$ with members from $D \cdot \frac{\pi}{2}$ and $x_n \to x$, $y_n \to y$. Now

$$|c(x) - c(y)| = \lim_n |\operatorname{Cos} \varphi(x_n) - \operatorname{Cos} \varphi(y_n)| \le \lim_n |x_n - y_n| = |x - y| < \varepsilon.$$

The function s is treated analogously. ∎

Due to Prop. 10 we already have

THEOREM 6. *For all $x, y \in \mathbb{R}$, there holds*

$$c(x - y) = c(x)c(y) + s(x)s(y).$$

THEOREM 7. *There holds*

$$\lim_{x \to 0+} \frac{s(x)}{x} = 1.$$

Proof. Due to Theorem 4, there holds

$$0 \le xc(x) \le s(x) \le x, \qquad x \in (0, \tfrac{\pi}{2}),$$

hence

$$c(x) \le \frac{s(x)}{x} \le 1, \qquad x \in (0, \tfrac{\pi}{2}).$$

It remains to note that $\lim_{x \to 0+} c(x) = c(0) = 1$. ∎

Due to Theorem 1, s and c coincide with sin and cos (ordinary sine and cosine), respectively. Hence, further on we only use the symbols sin and cos.

7. Inverse trigonometric functions

PROPOSITION 20. *The functions* sin *and* cos *from* $[0, \frac{\pi}{2}]$ *to* $[0, 1]$ *are bijections.*

Proof. The sine and cosine functions are strictly monotone when restricted to $D \cdot \frac{\pi}{2}$ (see Prop. 17). Assume that there are numbers $x, x' \in [0, \frac{\pi}{2}]$, $x < x'$ such that $\sin x = \sin x'$. Find $x_1, x_2 \in D \cdot \frac{\pi}{2}$ such that $x < x_1 < x_2 < x'$, then $\sin x \leq \sin x_1 < \sin x_2 \leq \sin x'$, a contradiction. The cosine function is strictly decreasing for a similar reason.

Surjectivity of $\sin|_{[0,\frac{\pi}{2}]} \colon [0, \frac{\pi}{2}] \to [0, 1]$ and $\cos|_{[0,\frac{\pi}{2}]} \colon [0, \frac{\pi}{2}] \to [0, 1]$ follows from the Bolzano–Cauchy theorem: a function f that is continuous in the interval $[a, b]$ attains all values between $f(a)$ and $f(b)$. ∎

PROPOSITION 21. *The functions* $\sin \colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ *and* $\cos \colon [0, \pi] \to [-1, 1]$ *are bijections.*

Proof. For the sine, the correspondences $[-\frac{\pi}{2}, 0] \leftrightarrow [-1, 0]$ and $[0, \frac{\pi}{2}] \leftrightarrow [0, 1]$ can be treated separately, due to $\sin(-x) = -\sin x$. The same about the cosine. ∎

Prop. 21 allows to define the inverse functions of the sine and cosine. Their domain is $[-1, 1]$, they are both continuous and strictly monotone.

The treatment of the homomorphism φ in the previous chapter left open the question whether all angles (elements of A) are images of some real number. This question will be resolved now.

PROPOSITION 22. *The restriction of the group homomorphism* $\varphi \colon \mathbb{R} \to A$, *the function* $\varphi|_{[0,2\pi)} \colon [0, 2\pi) \to A$, *is a group isomorphism.*

Proof. It suffices to prove that $\varphi|_{[0,\frac{\pi}{2}]} \colon [0, \frac{\pi}{2}] \to A_0$ is bijective. As sin and cos are bijective in this interval, φ is injective there. Now for any $\alpha = [\![(1, 0), (u, v)]\!]$ where $u^2 + v^2 = 1$ and $u, v \in [0, 1]$, we have $\sin x = u$ and $\cos x = \sqrt{1 - (\sin x)^2} = \sqrt{1 - u^2} = v$, hence $\varphi(x) = \alpha$. ∎

Prop. 22 allows to define the *size* of an angle α as $\left(\varphi|_{[0,2\pi]}\right)^{-1}(\alpha) \in [0, 2\pi)$.

ACKNOWLEDGEMENT. The author is grateful to the anonymous referee for the careful reading of the manuscript and detailed suggestions aimed at correcting errors as well as improving the exposition.

REFERENCES

[1] G. M. Fihtengol'c, *Course in Differential and Integral Calculus*, Vol. 2 [in Russian], FIZMATLIT, Moscow, 2001.

[2] *8th IMC 2001, July 19–25, Prague, Czech Republic. Second day*, `https://www.imc-math.org.uk/imc2001/prob_sol2.pdf`, Accessed: May 17, 2025.

[3] W. Rudin, *Principles of Mathematical Analysis*, McGraw Hill, 1976.

[4] M. Spivak, *Calculus*, Publish or Perish, Inc., Houston, 1994.

[5] H. E. Vaughan, *Characterization of the sine and cosine*, Amer. Math. Monthly **62** (1955), 707–713.

[6] F. Viète, *Francisci Vietae variorum de rebus mathematicis responsorum, liber VIII. Cuius praecipua capita sunt, De duplicatione cubi, & quadratione circuli. Quae claudit Procheirou, seu Ad vsum mathematici canonis methodica*, Apud Iamettium Mettayer, 1593.

Institute of Mathematics and Statistics, University of Tartu, Narva mnt 18, 51009 Tartu, Estonia

ORCID: 0000-0003-3230-3877

E-mail: indrek.zolk@ut.ee

Received: 20.05.2025, *in revised form* 19.08.2025

Accepted: 23.08.2025