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CONVEX LATTICE PENTAGON WITH THREE PAIRS OF PARALLEL SIDES AND DIAGONALS

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Abstract. This paper investigates convex lattice pentagons with at least three pairs (a_i, d_i), where $a_i \parallel d_i$, i.e., diagonals parallel to sides. Based on the given conditions, we will form a system of Diophantine equations whose solutions we seek within the set of natural numbers or positive rational numbers. To characterize all obtained convex lattice pentagons of minimal area, we will use the concept of integer unimodular transformations. Specifically, these transformations of the plane preserve the parallelism of lattice segments, the number of lattice points inside a convex lattice polygon and on its boundary, as well as its area. We will then determine the minimum area of the pentagon in each resulting class and identify the pentagon with the smallest diameter. Finally, we will determine all convex lattice pentagons in which three sides are respectively parallel to three diagonals.

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1. Introduction

Let a Cartesian coordinate system be defined in the plane. A point with integer coordinates is called a *lattice point*. A *lattice segment* is a line segment connecting two lattice points. A lattice segment is said to have lattice length k if it contains exactly $k-1$ lattice points in its interior. A *lattice vector* is the vector joining two lattice points. A polygon whose vertices are lattice points is called a *lattice polygon*. A lattice polygon in which all interior angles are less than $180°$ is called a *convex lattice polygon*.

The area of a lattice polygon can be determined using *Pick's Theorem* [2]:

$$S = i + \frac{b}{2} - 1,$$

where i is the number of lattice points inside the polygon and b is the number of lattice points on its boundary. It follows from Pick's Theorem that the minimal area of a lattice polygon (specifically, a lattice triangle) is $\frac{1}{2}$. A lattice triangle with area $\frac{1}{2}$ is called a *fundamental triangle*. For the area of triangle OAB, whose vertices are $O(0,0)$, $A(x_1, y_1)$ and $B(x_2, y_2)$, and which is positively oriented, we will also use the following formula:

$$S = \frac{1}{2}(x_1 y_2 - x_2 y_1).$$

For the study of the area of convex lattice polygons, a particularly important role is played by plane transformations that map a fundamental triangle to another fundamental triangle. Such transformations preserve the area of a convex lattice polygon and are fully determined by a 2×2 integer matrix whose determinant belongs to the set $\{-1, 1\}$.

Definition 1. A square matrix V is called *unimodular* if $\det V \in \{-1, 1\}$. A linear transformation is called *unimodular* if its matrix (in the standard basis of $\mathbb{R}^2$) is unimodular. A linear transformation is called *integer unimodular* if its matrix is both integer and unimodular.

Definition 2. The composition of a unimodular transformation and a translation is called a *unimodular affine transformation.* The composition of an integer unimodular transformation and an integer translation (i.e., translation by a lattice vector) is called an *integer unimodular affine transformation*, or *lattice equivalence.* Two lattice polygons are said to be *lattice equivalent* if there exists a lattice equivalence that maps one polygon onto the other.

The most important properties of integer unimodular transformations are stated in the following theorem.

Theorem 1. [3] *An integer unimodular transformation preserves the number of lattice points in a convex lattice polygon and on its boundary. The composition of two integer unimodular transformations is itself an integer unimodular transformation.*

Remark. The term *integer unimodular transformation* used in this paper is synonymous with *integral unimodular transformation*, as used in [3]. Both refer to linear transformations represented by matrices with integer entries and determinant ± 1.

The following theorem is an immediate consequence of Theorem 1.

Theorem 2. *Any two fundamental triangles are lattice-equivalent. In particular, every fundamental triangle is lattice-equivalent to the triangle with vertices* $(0,0)$, $(1,0)$, *and* $(0,1)$.

In the convex pentagon $A_1A_2A_3A_4A_5$, we denote in order: $\mathbf{a}_i = \overrightarrow{A_iA_{i+1}}$ as the side vectors, $a_i = A_iA_{i+1}$ as the sides, and a_i as their lengths, where $i \in \{1, 2, \ldots, 5\}$, $A_0 \equiv A_5$, $A_6 \equiv A_1$. To each side vector $\mathbf{a}_i = \overrightarrow{A_iA_{i+1}}$, we associate the *corresponding* diagonal vector $\mathbf{d}_i = \overrightarrow{A_{i-1}A_{i+2}}$, where $A_7 \equiv A_2$, and we denote d_i as the diagonal $A_{i-1}A_{i+2}$ and its length.

From the condition that $d_i \parallel a_i$, it follows that $\mathbf{d}_i = k_i\mathbf{a}_i$, for some $k_i > 0$, i.e.,

$$\mathbf{a}_{i-1} + \mathbf{a}_i + \mathbf{a}_{i+1} = k_i\mathbf{a}_i. \tag{1}$$

The positive number $k_i = \frac{d_i}{a_i}$ for which (1) holds is called the *parallelism coefficient* of the diagonal d_i and the side a_i.

If three vertices A, B, and C of a parallelogram $ABCD$ are lattice points, then the equality $\overrightarrow{AD} = \overrightarrow{BC}$ holds, from which it follows that the fourth vertex D is also a lattice point. Likewise, if the coordinates of points A, B, and C are rational, and r is a rational number such that $\overrightarrow{AD} = r\,\overrightarrow{BC}$, then the coordinates of point D are also rational. Based on this, we conclude that the parallelism coefficients k_i in equalities (1) are positive rational numbers.

Let the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ satisfy the relation $\overrightarrow{CD} = k\overrightarrow{AB}$ $(k > 0)$. Then, for positive numbers α and β, the four points $A_1(\alpha x_1, \beta y_1)$, $B_1(\alpha x_2, \beta y_2)$, $C_1(\alpha x_3, \beta y_3)$ and $D_1(\alpha x_4, \beta y_4)$ satisfy the equality $\overrightarrow{C_1D_1} = k\overrightarrow{A_1B_1}$, which is easily verified. This means that the line segments C_1D_1 and A_1B_1 are parallel, with the same parallelism coefficient k as the line segments CD and AB. The transformation that assigns the point (x, y) to the point $(\alpha x, \beta y)$ is called a *homothety with respect to the coordinate axes, with coefficients α and β.*

Lemma 1. [1] *In a convex lattice pentagon, the condition $d_i \parallel a_i$ cannot hold for all $i \in \{1, 2, 3, 4\}$.*

Proof. Assume that in a convex lattice pentagon, $d_i \parallel a_i$ for $i \in \{1, 2, 3, 4\}$, i.e., that there exist positive rational numbers k_i such that

$$\mathbf{a}_{i-1} + \mathbf{a}_i + \mathbf{a}_{i+1} = k_i\mathbf{a}_i, \quad i \in \{1, 2, 3, 4\}. \tag{2}$$

From this, for $i = 2$ and $i = 3$, we have, respectively,

$$\mathbf{a}_1 = k_2\mathbf{a}_2 - \mathbf{a}_2 - \mathbf{a}_3 = (k_2 - 1)\mathbf{a}_2 - \mathbf{a}_3,$$
$$\mathbf{a}_4 = k_3\mathbf{a}_3 - \mathbf{a}_2 - \mathbf{a}_3 = -\mathbf{a}_2 + (k_3 - 1)\mathbf{a}_3.$$

On the other hand, using the equality $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 = \mathbf{0}$ and the equalities (2) for $i = 4$ and $i = 1$, we further obtain that

$$\begin{aligned}\mathbf{a}_1 &= -\mathbf{a}_2 - (\mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5) = -\mathbf{a}_2 - k_4\mathbf{a}_4\\ &= -\mathbf{a}_2 - k_4(k_3\mathbf{a}_3 - \mathbf{a}_2 - \mathbf{a}_3) = (k_4 - 1)\mathbf{a}_2 - k_4(k_3 - 1)\mathbf{a}_3,\\ \mathbf{a}_4 &= -(\mathbf{a}_5 + \mathbf{a}_1 + \mathbf{a}_2) - \mathbf{a}_3 = -k_1\mathbf{a}_1 - \mathbf{a}_3\\ &= -k_1(k_2\mathbf{a}_2 - \mathbf{a}_2 - \mathbf{a}_3) - \mathbf{a}_3 = -k_1(k_2 - 1)\mathbf{a}_2 + (k_1 - 1)\mathbf{a}_3.\end{aligned}$$

By equating the corresponding coefficients of the vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ in the obtained expressions for $\mathbf{a}_1$ and $\mathbf{a}_4$, from the previous four equalities we get that $k_2 - 1 = k_4 - 1$,

$$1 = k_4(k_3 - 1), \tag{3}$$

$$1 = k_1(k_2 - 1), \tag{4}$$

and $k_3 - 1 = k_1 - 1$. It follows that $k_4 = k_2$ and $k_3 = k_1$, so by substituting into equation (3), we obtain $1 = k_2(k_1 - 1)$. From this and equation (4), it follows that $k_1 = k_2$, which implies $k_1 = k_2 = k_3 = k_4 = k$, and that $k^2 - k - 1 = 0$, i.e.

$$k_1 = k_2 = k_3 = k_4 = k = \frac{1 + \sqrt{5}}{2}.$$

Since $\frac{1+\sqrt{5}}{2}$ is an irrational number, we obtain a contradiction. ∎

It follows from the previous lemma that the following statement holds:

THEOREM 3. [1] *In a convex lattice pentagon, at most three relations of the form $d_i \parallel a_i$ can hold.*

Figure 1 shows an example of a convex lattice pentagon in which the relations $d_1 \parallel a_1$, $d_2 \parallel a_2$ and $d_4 \parallel a_4$ hold.

LEMMA 2. *If in a convex pentagon it holds that $\mathbf{d}_1 = k_1\mathbf{a}_1$ and $\mathbf{d}_2 = k_2\mathbf{a}_2$, $(k_1, k_2 > 0)$, then $k_1 > 1$ and $k_2 > 1$.*

Proof. Let $A_1A_2A_3A_4A_5$ be a convex pentagon such that $\mathbf{d}_1 = k_1\mathbf{a}_1$ and $\mathbf{d}_2 = k_2\mathbf{a}_2$, and let D be the point such that $A_1A_2A_3D$ is a parallelogram. Due to the convexity of the given pentagon, it follows that point A_4 lies on the extension of the segment A_1D beyond the vertex D, and that point A_5 lies on the extension of the segment A_3D beyond D. Thus, from the equalities $\mathbf{d}_1 = k_1\mathbf{a}_1$ and $\mathbf{d}_2 = k_2\mathbf{a}_2$, we obtain that $d_1 > a_1$ and $d_2 > a_2$, that is, $k_1 > 1$ and $k_2 > 1$. ■

2. Main results

In this section, we consider convex lattice pentagons under given parallelism conditions $d_i \parallel a_i$ between certain diagonals and sides. From Theorem 3, it follows that at most three such conditions can hold in a convex lattice pentagon. We now examine convex lattice pentagons in which exactly three such parallelism conditions hold.

Let a convex lattice pentagon satisfy three relations of the form $d_i \parallel a_i$. Since a translation by a lattice vector is an isometry that maps lattice points to lattice points, we may assume (without loss of generality) that one vertex of the pentagon lies at the origin. Let $A_2 = (0, 0)$. There are now essentially two distinct cases to consider.

2.1 Convex lattice pentagons with given conditions $\boldsymbol{d}_1 \parallel \boldsymbol{a}_1$, $\boldsymbol{d}_2 \parallel \boldsymbol{a}_2$ and $\boldsymbol{d}_4 \parallel \boldsymbol{a}_4$

Let p, q be natural numbers such that $(p, q) = 1$ and $p > q \geq 1$.

Let $\mathcal{M}_1$ denote the set of all convex lattice pentagons of minimal area that satisfy the conditions $d_1 \parallel a_1$, $d_2 \parallel a_2$ and $d_4 \parallel a_4$. Also, let $\mathcal{P}_1(p, q)$ denote the set of all convex lattice pentagons with parallelism coefficients $k_1 = k_2 = \frac{p}{q}$ and $k_4 = \frac{q}{p-q}$ and let $\mathcal{M}_1(p, q)$ be the subset of such pentagons with minimal area.

THEOREM 4. 1) *Every element of $\mathcal{M}_1$ has area $5/2$, parallelism coefficients $k_1 = k_2 = 2$ and $k_4 = 1$, and is lattice-equivalent to a convex lattice pentagon in which $d_i \parallel a_i$ for $i \in \{1, 2, 4\}$, as shown in Figure 1.*

2) *For every pair of natural numbers p and q such that $(p, q) = 1$ and $p > q \geq 1$, there exists a convex lattice pentagon with parallelism coefficients $k_1 = k_2 = \frac{p}{q}$ and $k_4 = \frac{q}{p-q}$, i.e., the set $\mathcal{P}_1(p, q)$ is non-empty.*

3) *Every element of* $\mathcal{M}_1(p,q)$ *has area* $\frac{1}{2}(p^2+q^2)$. *Every element of* $\mathcal{P}_1(p,q)$ *can be obtained from some element of* $\mathcal{M}_1(p,q)$ *by applying a homothety with integer coefficients along the coordinate axes, followed by a translation by a lattice vector.*

4) *The minimal diameter among the elements of* $\mathcal{M}_1$ *is* $\sqrt{5}$, *and it is attained by the pentagon shown in Figure 1.*

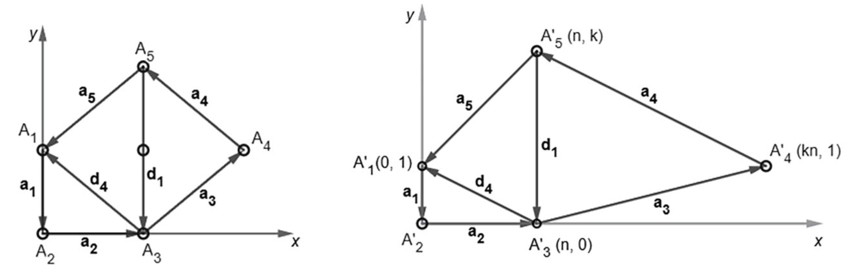

Figure 1. A convex lattice pentagon in which $d_i \parallel a_i$, $i \in \{1,2,4\}$

Proof. 1) Based on (1), there exist positive rational numbers k_1, k_2, and k_4 such that the following holds (Figure 1):

$$\mathbf{a}_5+\mathbf{a}_1+\mathbf{a}_2=k_1\mathbf{a}_1, \tag{5}$$

$$\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3=k_2\mathbf{a}_2, \tag{6}$$

$$\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5=k_4\mathbf{a}_4. \tag{7}$$

From (5) and (6), we obtain that

$$\mathbf{a}_5=k_1\mathbf{a}_1-\mathbf{a}_1-\mathbf{a}_2, \mathbf{a}_3=k_2\mathbf{a}_2-\mathbf{a}_1-\mathbf{a}_2.$$

Now, since $\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5=\mathbf{0}$, it follows that

$$\mathbf{a}_4=-(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)-\mathbf{a}_5=-k_2\mathbf{a}_2-k_1\mathbf{a}_1+\mathbf{a}_1+\mathbf{a}_2,$$

By substituting these values for $\mathbf{a}_3, \mathbf{a}_4$ and $\mathbf{a}_5$ into (7), we get

$$\begin{aligned}k_2\mathbf{a}_2-\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_1+\mathbf{a}_2-k_2\mathbf{a}_2-k_1\mathbf{a}_1+k_1\mathbf{a}_1-\mathbf{a}_1-\mathbf{a}_2\\ =k_4(\mathbf{a}_1+\mathbf{a}_2-k_1\mathbf{a}_1-k_2\mathbf{a}_2).\end{aligned}$$

By simplifying, we obtain in sequence that:

$$\begin{gathered}(k_1k_4-k_4-1)\mathbf{a}_1+(k_2k_4-k_4-1)\mathbf{a}_2=0,\\ k_1k_4-k_4-1=0 \quad \text{and} \quad k_2k_4-k_4-1=0.\end{gathered}$$

By subtracting the last two equations, we obtain $k_4(k_1-k_2)=0$. Since k_4 is a positive number, it follows that $k_1=k_2$. Therefore, the following equalities hold:

$$k_1=1+\frac{1}{k_4}=k_2.$$

Let us denote $k_1 = k_2 = k$ and $k_4 = \frac{1}{k-1}$. According to Lemma 2, we have $k > 1$. We now construct a convex pentagon $A'_1A'_2A'_3A'_4A'_5$ with these parallelism coefficients, where three of its vertices are $A'_1 = (0,1)$, $A'_2 = (0,0)$ and $A'_3 = (1,0)$. From the equalities $\overrightarrow{A'_1A'_4} = k\overrightarrow{A'_2A'_3}$ and $\overrightarrow{A'_3A'_5} = k\overrightarrow{A'_2A'_1}$, it follows that the remaining two vertices are $A'_4 = (k,1)$ and $A'_5 = (1,k)$. At this point, we have

$$\mathbf{a}_4 = \overrightarrow{A'_4A'_5} = (1-k)\mathbf{i} + (k-1)\mathbf{j} = (k-1)(-\mathbf{i}+\mathbf{j}) = (k-1)\overrightarrow{A'_3A'_1} = (k-1)\mathbf{d}_4,$$

i.e. $\mathbf{d}_4 = \frac{1}{k-1}\mathbf{a}_4 = k_4\mathbf{a}_4$. Thus, in the convex pentagon whose vertices are

(8) $\quad A'_1 = (0,1),\ \ A'_2 = (0,0),\ \ A'_3 = (1,0),\ \ A'_4 = (k,1),\ \ A'_5 = (1,k) \quad (k > 1),$

the coefficients of parallelism are $k_1 = k_2 = k$ and $k_4 = \frac{1}{k-1}$.

The lattice point $D = (1,1)$, by Lemma 2, lies in the interior of the convex pentagon $A'_1A'_2A'_3A'_4A'_5$. For $k = 2$, we obtain a convex lattice pentagon $A_1A_2A_3A_4A_5$ with vertices $A_1(0,1)$, $A_2(0,0)$, $A_3(1,0)$, $A_4(2,1)$ and $A_5(1,2)$ in which the coefficients of parallelism are $k_1 = k_2 = 2$ and $k_4 = 1$ (see Figure 1). The area of this pentagon, according to Pick's Theorem, is given by

$$S = i + \frac{b}{2} - 1 = 1 + \frac{5}{2} - 1 = \frac{5}{2},$$

which is the minimal possible area of a convex lattice pentagon.

Every such convex lattice pentagon with area $\frac{5}{2}$ is lattice-equivalent to the pentagon $A_1A_2A_3A_4A_5$ shown in Figure 1. Indeed, if $B_1B_2B_3B_4B_5$ is a convex lattice pentagon of area $\frac{5}{2}$, it follows from the above that its coefficients of parallelism are $k_1 = k_2 = 2$ and $k_4 = 1$. By Theorems 1 and 2, there exists a lattice equivalence mapping the triangle $A_1A_2A_3$ onto the triangle $B_1B_2B_3$. The points B_4 and B_5 are uniquely determined by the conditions $\overrightarrow{B_1B_4} = 2\overrightarrow{B_2B_3}$ and $\overrightarrow{B_3B_5} = 2\overrightarrow{B_2B_1}$.

2) Since the coefficients of parallelism in a convex lattice pentagon are positive rational numbers, we have

$$k = \frac{p}{q}, \qquad p, q \in \mathbb{Z}, \quad (p,q) = 1, \quad p > q \geq 1.$$

If we substitute this value of k into (8) and multiply all coordinates of the resulting pentagon by q, we obtain a convex lattice pentagon with vertices

(9) $\qquad A_1(0,q), \quad A_2(0,0), \quad A_3(q,0), \quad A_4(p,q), \quad A_5(q,p),$

and the coefficients of parallelism are $k_1 = k_2 = \frac{p}{q}$ and $k_4 = \frac{q}{p-q}$. Its area is equal to the sum of areas of the triangles $A_2A_3A_4$, $A_2A_4A_5$ and $A_2A_5A_1$, and is given by

$$S = \frac{1}{2}(q^2 - 0 + p^2 - q^2 + q^2 - 0) = \frac{1}{2}(p^2 + q^2).$$

Since $p > q \geq 1$, it is clear that this area is minimal when $p = 2$ and $q = 1$, i.e., for the convex lattice pentagon shown in Figure 1.

3) Let a lattice unimodular transformation be defined by the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \quad |ad - bc| = 1.$$

This transformation maps the vertices of the convex lattice pentagon given in (9) to the vertices

(10) $B_1(bq, dq), \ \ B_2(0,0), \ \ B_3(aq, cq), \ \ B_4(ap+bq, cp+dq), \ \ B_5(aq+bp, cq+dp),$

which form a convex lattice pentagon of the same area $S = \frac{1}{2}(p^2 + q^2)$, and with the same coefficients of parallelism $k_1 = k_2 = \frac{p}{q}$ and $k_4 = \frac{q}{p-q}$.

4) Let us show that the minimal diameter of the pentagon given in (10) is equal to $\sqrt{5}$. We will prove that

$$\max\{B_1B_4, B_3B_5, B_2B_4\} \geq \sqrt{5}.$$

Assume the contrary. Then $B_1B_4{}^2 < 5$ and $B_3B_5{}^2 < 5$, from which we successively obtain:

$$(ap)^2 + (cp)^2 = p^2(a^2 + c^2) < 5 \quad \text{and} \quad (bp)^2 + (dp)^2 = p^2(b^2 + d^2) < 5.$$

Since $p \geq 2$, it follows that

$$a^2 + c^2 \leq 1 \quad \text{and} \quad b^2 + d^2 \leq 1.$$

Since $|ad - bc| = 1$, it cannot be $a^2 + c^2 = 0$ or $b^2 + d^2 = 0$, and thus we must have:

$$a^2 + c^2 = 1 \quad \text{and} \quad b^2 + d^2 = 1. \tag{11}$$

Now, using the identity

$$(ab + cd)^2 + (ad - bc)^2 = (a^2 + c^2)(b^2 + d^2), \tag{12}$$

it follows that $ab + cd = 0$, and using (11) we obtain

$$\begin{aligned} B_2B_4{}^2 &= (ap + bq)^2 + (cp + dq)^2 = p^2(a^2 + c^2) + q^2(b^2 + d^2) + 2pq(ab + cd) \\ &= p^2 + q^2 \geq 5, \end{aligned}$$

i.e., $B_2B_4 = \sqrt{5}$. This is a contradiction. ∎

2.2 Convex lattice pentagons with given conditions $\boldsymbol{d}_1 \parallel \boldsymbol{a}_1$, $\boldsymbol{d}_2 \parallel \boldsymbol{a}_2$ and $\boldsymbol{d}_3 \parallel \boldsymbol{a}_3$

Let p, q be natural numbers such that $(p, q) = 1$ and $p > q \geq 1$.

Let $\mathcal{M}_2$ denote the set of all convex lattice pentagons of minimal area that satisfy the conditions $d_1 \parallel a_1$, $d_2 \parallel a_2$ and $d_3 \parallel a_3$. Also, let $\mathcal{P}_2(p, q)$ denote the set of all convex lattice pentagons with parallelism coefficients $k_1 = k_3 = \frac{p}{q}$ and $k_2 = \frac{p+q}{p}$, and let $\mathcal{M}_2(p, q)$ be the subset of such pentagons with minimal area.

THEOREM 5. 1) *Every element of* $\mathcal{M}_2$ *has area* 4*, parallelism coefficients* $k_1 = k_3 = 2$ *and* $k_2 = \frac{3}{2}$*, and is lattice-equivalent to a convex lattice pentagon in which* $d_i \parallel a_i$ *for* $i \in \{1, 2, 3\}$*, as shown in Figure 2.*

2) *For every pair of natural numbers* p *and* q *such that* $(p, q) = 1$ *and* $p > q \geq 1$*, there exists a convex lattice pentagon with parallelism coefficients* $k_1 = k_3 = \frac{p}{q}$ *and* $k_2 = \frac{p+q}{p}$*, i.e., the set* $\mathcal{P}_2(p, q)$ *is non-empty.*

3) *Every element of* $\mathcal{M}_2(p, q)$ *has area* $\frac{1}{2}p(p + 2q)$*. Every element of* $\mathcal{P}_2(p, q)$ *can be obtained from some element of* $\mathcal{M}_2(p, q)$ *by applying a homothety with integer coefficients along the coordinate axes, followed by a translation by a lattice vector.*

4) *The minimal diameter among the elements of* $\mathcal{M}_2$ *is* $\sqrt{10}$*, and it is attained by the pentagon shown in Figure 2.*

5) *Every convex lattice pentagon with at least three pairs of parallel sides and diagonals belongs to exactly one of the sets* $\mathcal{P}_1(p, q)$ *or* $\mathcal{P}_2(p, q)$*.*

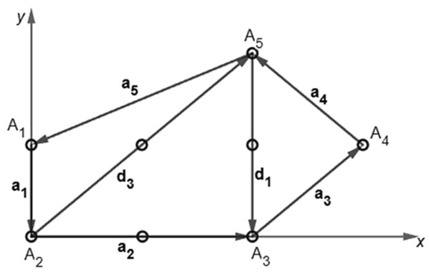

Figure 2. A convex lattice pentagon in which $d_i \parallel a_i$, $i \in \{1, 2, 3\}$

Proof. 1) Based on (1), there exist positive rational numbers k_1, k_2 and k_3 such that (see Figure 2):

$$\mathbf{a}_5 + \mathbf{a}_1 + \mathbf{a}_2 = k_1\mathbf{a}_1, \tag{13}$$

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = k_2\mathbf{a}_2, \tag{14}$$

$$\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = k_3\mathbf{a}_3. \tag{15}$$

Since $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 = \mathbf{0}$, by adding equalities (13) and (15), we obtain

$$k_1\mathbf{a}_1 + k_3\mathbf{a}_3 - \mathbf{a}_2 = 0. \tag{16}$$

From (14), it follows that $\mathbf{a}_3 = k_2\mathbf{a}_2 - \mathbf{a}_1 - \mathbf{a}_2$, and by substituting into (16), we obtain

$$k_1\mathbf{a}_1 + k_3(k_2\mathbf{a}_2 - \mathbf{a}_1 - \mathbf{a}_2) - \mathbf{a}_2 = 0,$$
$$(k_1 - k_3)\mathbf{a}_1 + (k_2k_3 - k_3 - 1)\mathbf{a}_2 = 0,$$

from which we obtain $k_1 - k_3 = 0$ and $k_2k_3 - k_3 - 1 = 0$. It follows that $k_3 = k_1$ and $k_2k_1 = k_1 + 1$, i.e.,

$$k_2 = 1 + \frac{1}{k_1}.$$

Let $k_1 = k$. Then $k_2 = 1 + \frac{1}{k}$ and $k_3 = k$. By Lemma 2, we have $k > 1$. Let us construct a convex pentagon $A_1'A_2'A_3'A_4'A_5'$, with these parallelism coefficients, and with three of its vertices given by $A_1'(0,1)$, $A_2'(0,0)$, $A_3'(1,0)$. From the equalities $\overrightarrow{A_1'A_4'} = (1+\frac{1}{k})\overrightarrow{A_2'A_3'}$ and $\overrightarrow{A_3'A_5'} = k\overrightarrow{A_2'A_1'}$ we obtain that the remaining two vertices are $A_4' = (1+\frac{1}{k}, 1)$, $A_5' = (1,k)$. We have

$$\mathbf{a}_3 = \overrightarrow{A_3'A_4'} = \frac{1}{k}\mathbf{i} + \mathbf{j} = \frac{1}{k}(\mathbf{i} + k\mathbf{j}) = \frac{1}{k}\overrightarrow{A_2'A_5'} = \frac{1}{k}\mathbf{d}_3,$$

that is, $\mathbf{d}_3 = k\mathbf{a}_3 = k_3\mathbf{a}_3$. Therefore, in the convex pentagon with vertices

$$(17) \qquad A_1'(0,1),\ A_2'(0,0),\ A_3'(1,0),\ A_4' = (1+\tfrac{1}{k}, 1),\ A_5' = (1,k) \quad (k>1),$$

the parallelism coefficients are $k_1 = k_3 = k$ and $k_2 = 1 + \frac{1}{k}$.

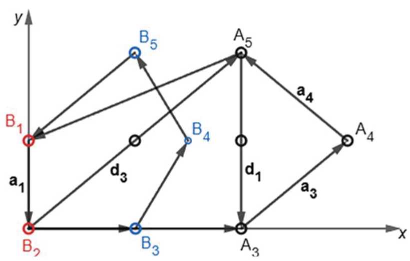

Figure 3. A convex lattice pentagon in which $d_i \parallel a_i$, $i \in \{1,2,3\}$

For $k = 2$, we obtain a convex pentagon $B_1B_2B_3B_4B_5$ with vertices $B_1(0,1)$, $B_2(0,0)$, $B_3(1,0)$, $B_4(\frac{3}{2},1)$, $B_5(1,2)$ and parallelism coefficients $k_1 = k_3 = 2$ and $k_2 = \frac{3}{2}$ (see Figure 3). By multiplying the first coordinates of this pentagon by 2, we obtain a convex lattice pentagon $B_1B_2A_3A_4A_5$ with the same parallelism coefficients and vertices $B_1(0,1)$, $B_2(0,0)$, $A_3(2,0)$, $A_4(3,1)$ and $A_5(2,2)$. The area of this pentagon, according to Pick's Theorem, is given by

$$S = i + \frac{b}{2} - 1 = 2 + \frac{6}{2} - 1 = 4,$$

which is the minimal possible area of a convex lattice pentagon in which $d_1 \parallel a_1$, $d_2 \parallel a_2$ and $d_3 \parallel a_3$ (see Figure 2).

2) Since the coefficients of parallelism in a convex lattice pentagon are positive rational numbers greater than 1, we have

$$k = \frac{p}{q}, \qquad p,q \in \mathbb{Z}, \quad (p,q) = 1, \quad p > q \geq 1.$$

If we substitute this value of k into (17) and multiply the first coordinates of the resulting pentagon by p, and the second coordinates by q, we obtain a convex lattice pentagon with vertices

$$(18) \qquad A_1(0,q),\ A_2(0,0),\ A_3(p,0),\ A_4(p+q,q),\ A_5(p,p)$$

and the coefficients of parallelism are $k_1 = k_3 = \frac{p}{q}$ and $k_2 = \frac{p+q}{p}$. Its area is equal to the sum of areas of the triangles $A_2A_3A_4$, $A_2A_4A_5$ and $A_2A_5A_1$, and is given by

$$S = \frac{1}{2}(pq - 0 + (p+q)p - pq + pq - 0) = \frac{1}{2}p(p+2q).$$

It is clear that this area will be minimal when $p = 2$ and $q = 1$, i.e., $S = 4$ for a convex lattice pentagon in which $d_1 \parallel a_1$, $d_2 \parallel a_2$ and $d_3 \parallel a_3$ (see Figure 2).

3) Let a lattice unimodular transformation be defined by the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \quad |ad - bc| = 1.$$

This transformation maps the vertices of the convex lattice pentagon given in (18) to the vertices $B_1(bq, dq)$, $B_2(0,0)$, $B_3(ap, cp)$, $B_4(ap + aq + bq, cp + cq + dq)$ and $B_5(ap + bp, cp + dp)$, which form a convex lattice pentagon of the same area $S = \frac{1}{2}p(p+2q)$, and with the same coefficients of parallelism $k_1 = k_3 = \frac{p}{q}$ and $k_2 = \frac{p+q}{p}$.

4) Let us show that the minimal diameter of the pentagon $B_1B_2B_3B_4B_5$ is equal to $\sqrt{10}$. We will prove that

$$\max\{B_1B_4, B_2B_4, B_1B_3\} \geq \sqrt{10}.$$

Assume the contrary. Then $B_1B_4{}^2 < 10$, from which we get $(ap+aq)^2+(cp+cq)^2 \leq 9$, or equivalently, $(a^2 + c^2)(p + q)^2 \leq 9$. Since $a^2 + c^2 \geq 1$, $p \geq 2$ and $q \geq 1$, it follows that $a^2 + c^2 = 1$, $p = 2$, $q = 1$. Thus, the vertices of the pentagon are

$$B_1(b,d), \quad B_2(0,0), \quad B_3(2a,2c), \quad B_4(3a+b, 3c+d) \text{ and } B_5(2a+2b, 2c+2d).$$

Let us denote $m = ab+cd$. Now, from $B_2B_4{}^2 < 10$, we obtain $(3a+b)^2+(3c+d)^2 \leq 9$, or equivalently $9(a^2 + c^2) + b^2 + d^2 + 6m \leq 9$. Since $a^2 + c^2 = 1$, it follows that $b^2 + d^2 \leq -6m$, and therefore $m < 0$.

Furthermore, from $B_1B_3{}^2 < 10$, we obtain $(2a - b)^2 + (2c - d)^2 \leq 9$, or equivalently $4(a^2 + c^2) + b^2 + d^2 - 4m \leq 9$. Since $a^2 + c^2 = 1$, it follows that $b^2 + d^2 \leq 5 + 4m$, i.e.,

$$1 \leq b^2 + d^2 \leq 5 + 4m. \tag{19}$$

It follows that $1 \leq 5 + 4m$, i.e., $-1 \leq m$. Hence, $-1 \leq m < 0$, so $m = -1$. For $m = -1$, from (19) we obtain $b^2 + d^2 = 1$. Equation (12) now becomes $m^2 + 1 = 1$, which implies $m = 0$. A contradiction. ■

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