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SOLVING INEQUALITIES IN PRIMARY SCHOOL

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Abstract. When sums and differences with a variable component are compared with a fixed number, simple inequalities suitable for solving in the set N of natural numbers are obtained. Solving of these inequalities bases upon the properties of such expressions to increase or decrease, depending on the change of values of their components.

To establish the meaning of an inequality, children have to be stimulated to see it as a representation of a whole set of numerical relations, some of which are true and some false. Then, the search for those values of the variable component for which these relations are true is the procedure of solving an inequality.

Thinking of prerequisite knowledge and skills for this procedure, some exercises have to be planed that will help children assimilate the meaning of some operative terms: expression, value of an expression, to take value, etc., as well as to instruct them in using first set-theoretical notations properly.

Not counting general observations, the entire text of this paper resembles a concrete elaboration, appropriate for school practice.

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Key words and phrases: Expression, Value of an expression, Variable summand (minuend) (subtrahend), Inequality, Set of solutions.

1. Introduction

The use of letters in elementary school arithmetic is aimed at an early development of the idea of variable and the selection of items to be included in school contents as well as the ways of their elaboration have to be subjected to this important didactical task. A literal expression or a literal relation have to be seen as representing a whole set of particular cases in both ways—by generalizing the particular and by specifying the general. And just the primary grades seem to be the right place to commence with a proper realization of this task.

The following four areas for the use of letters can be singled out:

1. Solving of equations,
2. Solving of inequalities,
3. Symbolic expressions of the rules (laws) of arithmetic,
4. Composing of literal expressions representing the laws of correspondence of two quantities (germs of the idea of function).

In this paper we focus our attention on the second of these areas, suggesting a procedure for solving simple inequalities based on variation of sums and differences depending on changes of value of their components ([7], [8]). This approach of elaboration of inequalities has been widely adopted in Serbian primary schools and the corresponding practice shows no difficulties in realization of this theme, (what several of our interviews with teachers also confirm).

Let us notice that inequalities are a highly motivating topic that helps children to develop the idea of variable—where each of them comprises a whole set of numerical relations, some of which are true and some that are false. In contrast with the case of equations, here the searched values of the variable constitute a whole set and not only a single unknown number.

In the practice that still continues to exist, the following two unfortunate circumstances are encountered—inequalities to be solved are unreasonably complex or they are wrongly transformed into equivalent ones, failing to see that such equivalences, though valid for the set **R** of real numbers (or **Q** the set of rational numbers), are transformations which do not hold in the case of the set **N** of natural numbers. For instance, in **N**, the inequality $x : 11 < 10$ is not equivalent to the inequality $x < 110$ and, when solved in **N**, it has $\{11, 22, \ldots, 99\}$ for the set of its solutions. Avoiding such awkward cases of inequalities that would complicate a normal elaboration of this theme, we confine our considerations to the following six types of inequalities:

$$a + x < b, \quad a + x > b, \quad x - a < b, \quad x - a > b, \quad a - x < b, \quad a - x > b.$$

Although simple, these types are sufficient to serve well the fulfillment of the mentioned didactical task.

There exists a long series of papers concerning Early Algebra and they manifest a variation of perspectives: building on child?s initial understanding ([1], [2], [3]), projecting a schematic-theoretic view ([14]), promoting transition from arithmetical to algebraic thinking ([4], [5], [6]), treating algebraic thinking as a sign-mediated cognitive praxis ([10]), etc.

As for the first of the above ways, we base solving of equations on interdependence of arithmetic operations ([7], [8]) and not on logical equivalence of conditions that are produced by "adding equals to equals and subtracting equals from equals" ([11]).

Here and in general, we follow Skemp's ideas of relational understanding ([12], [13]), which can result only after a thorough didactical analysis of the matter under consideration and by a proper didactical shaping of that matter.

2. Looking at the idea of variable

The use of letters as notations for an unknown can be traced back to the Grecian mathematics. With the Vieta's *logistica speciosa* (François Viète, 1540–1603) the letters had been used to denote "species of numbers" and when Gallilleo's contemporaries had used them to compose the formula expressing symbolically his Law of Vertical Projectile that was containing "everything"—the height which the projectile reaches, the time when it happens and the time when the projectile falls back to the ground, it was so much spectacular and inspiring to follow that various relationships between magnitudes and between geometric elements were expressed in that symbolic way. The Vieta's use of letters introduced the idea of variable into mathematics effecting its very intensive development (creation of Analytic Geometry, Calculus, etc.).

On the other hand, in the countries which had enlightened approach to education, the efforts to introduce the idea of variable and the concept of function into the school contents of mathematics can be traced back to the whole course of the 19th century. In the beginning of the 20th century, under the guidance of the great mathematician and educator Felix Klein (1849–1925), an important innovatory program (known as Merano Program, Merano, 1905) was created and which had been influencing the teaching of school mathematics throughout the first half of that century. In particular, Klein underlined the fact that function is the dominating concept of modern mathematics and he pleaded for such elaborations of the school contents that would help the development of "functional thinking". He wrote that it would not be enough to give the definition of function, but that that concept should penetrate entire teaching of mathematics in schools.

In the period of "New Maths", an intensive use of letters and correspondences manifesting the idea of function had been abundant in school mathematics at all levels. Then, the concept of function did indeed penetrate all contents of school mathematics but, dare we say it, there was a serious lack of right didactical elaborations. And as the teaching practice shows it clearly, general concepts have first to be used implicitly and in fragments before they gain a full meaning. Such a task requires a series of proper didactical procedures and we hope that our paper is a modest link in such a chain.

As one can easily observe it, in the older mathematical writings the term "idea of variable" is used instead of the term "concept of variable". The reason for it can be seen in the fact that without serviceable tools of set theory such a conceptualization was not possible. Based on the concept of set, the definition of variable goes as follows:

Given a letter, say, x and a non-empty set, say, S, when x denotes any element of the set S, then the letter x is called *variable in the set* S and the set S is called *universal set of the variable* x.

Though true, it would be trivial to insist that any letter can be a variable and any non-empty set its universal set. Instead of it, fixed notations have traditionally been used to denote some important sets in mathematics and variables in them:

$\mathbf{N}$ – the set of natural numbers, n – variable in $\mathbf{N}$,

$\mathbf{Q}$ – the set of rational numbers, q – variable in $\mathbf{Q}$,

$\mathbf{R}$ – the set of real numbers, x – variable in $\mathbf{R}$,

$\mathbf{C}$ – the set of complex numbers, z – variable in $\mathbf{C}$, etc.

When two or more variables are considered at the same time, two or more different letters are used to denote them. For example, in the case of the set $\mathbf{N}$, the letters $l, m, n, \ldots$ are used, in the case of the set $\mathbf{R}$, the letters $x, y, z, \ldots$, etc.

Let us also remark that the letter x is often used to denote a variable irrespective of its universal set.

When children use letters, then they denote variables in the set of numbers with which they have been already acquainted. Thus, for the second graders such a set is the block $\mathbf{N}_{100}$ of numbers up to 100, for the third graders the block $\mathbf{N}_{1000}$ of numbers up to 1000 and for the fourth graders the set $\mathbf{N}$ of all natural numbers. By keeping all essential considerations inside the limits of those sets and by adopting respective notations, the teacher helps children to develop a feeling for such frames to which variation of a variable is bounded.

3. Why exemplification is needed

But what constitutes mathematics is not the subject matter, but a particular kind of knowledge about it.

R. Skemp

It is extremely simple to formulate mathematically the ways of solution of inequalities that are considered in this paper. For example, the inequality $a-x<b$ is solved by solving the equation: $a-x=b$, $x=a-b$. Then, using the fact that the difference $a-x$ decreases when x increases, the solutions (in $\mathbf{N}$) are all natural numbers larger than $a-b$ and not exceeding the minuend a. In this way the skeleton of the argument is fleshed out, but all subtleties providing the way of understanding that argument by a child are missing.

Prior to elaboration of this theme, children ought to develop some skills in dealing with numerical expressions without an automatic stimulus to calculate their values. But with these skills alone, literal expressions as, for example, $x+3$ or literal relations as, for example, $x+3<15$ will have no a priori meaning to them. Thus, a number of deliberately designed didactical procedures are needed and, instead of describing them, it is more reasonable to exemplify them by the pieces of text that refer directly to children (and that look like they were torn from a child's textbook). In this paper, these pieces are separated from the rest of the text in an easily noticeable way. Let us also add that without pointing out what children have learned before and how they have done it, elaboration of a topic or study of it must necessarily be incomplete.

Let us recall some basic facts that teachers should always have in mind. Each numerical expression represents a unique number. But its form may be not much informative as it is the decimal notation representing that same number. To calculate the value of a numerical expression means exactly to find such its decimal notation. And while each numerical expression has its unique value, a variable or a literal expression takes values. This way of saying things helps children to acquire variable quantities as being always changing. Thus, a correct assimilation of the involved terms is essential and we will also sketch here an elaboration that resembles classroom practice.

Connected with the solving of inequalities, set theoretical notations appear for the first time as being really needed. The use of brackets to denote sets as organized wholes, modified "e" to denote relationship between an element and a set, ellipsis to suggest an ordering of numbers, are little bits of technique that also require some careful elaboration. Finally, let us say that teachers have to be aware of the fact that brackets are syntactic signs and that they should not be placed around all sorts of strange things.

Before proceeding with the direct elaboration of inequalities, we will sketch the ways of forming prerequisite knowledge and skills to that theme.

4. Expressions and their values

Loosely speaking, expressions are collections of symbols obtained when numbers and letters are combined with the operation signs. For children "expression" is a general term for sums, differences, products and quotients. Before starting to assimilate the meaning of that term, children should be assigned a number of exercises as, for instance, the following are:

(i) $3 \cdot 9 + 15 : 3$ is ________ (a sum). The first summand is ________ $(3 \cdot 9)$ and the second ________ $(15 : 3)$.

$3 \cdot 9$ is ________ (a product) and $15 : 3$ ________ (a quotient).

(ii) $(22 - 2) : 5$ is ________ (a quotient). The dividend is ________ $(22 - 2)$ and the divisor is ________ (5).

$22 - 2$ is ________ (a difference). Etc.

(The answers expected to be given by children are put in brackets).

At this level the meaning of a general term is formed by an active use of a number of suitable examples or by means of more specific concepts that it includes. Thus, in this case we can proceed as follows:

Sums as, for example,

$$7 + 18, \quad 3 \cdot 9 + 2 \cdot 7, \quad 36 : 9 + 4, \quad \ldots,$$

differences as, for example,

$$75 - 28, \quad 7 \cdot 9 - 3 \cdot 7, \quad 57 - 99 : 3, \quad \ldots,$$

products as, for example,

$$15 \cdot 3, \quad 4 \cdot (12 \cdot 8), \quad (27 : 3) \cdot 5, \quad \ldots,$$

quotients as, for example,

$$63:7,\quad (17-2):5,\quad (12\cdot 2):4,\quad \ldots,$$

are called, in a single word, *expressions*.

Calculating as in the following cases:

(i) $3\cdot 9+2\cdot 7=27+14=41$, (ii) $57-99:3=57-33=24$,

(iii) $4\cdot(12+8)=4\cdot 20=80$, (iv) $(12\cdot 2):4=24:4=6$,

we find that:

(i) The expression $3\cdot 9+2\cdot 7$ represents the number 41. Then we say that the number 41 is *the value of the expression* $3\cdot 9+2\cdot 7$.

(ii) The expression $57-99:3$ represents the number ______ (24). Then we say that the number ______ (24) is the value of the expression $57-99:3$.

(iii) The expression $4\cdot(12+8)$ has the value ______ (80).

(iv) The value of the expression $(12\cdot 2):4$ is ______ (6). Etc.

Completing these and similar sentences (by filling in the appropriate answers), children learn actively the meaning of the terms "expression" and "value of expression".

5. Evaluation of literal expressions

Children meet the first case of evaluation of literal expressions when they check an equation for the right solution. For example, the first graders solve such simple equations as the following one:

$$x+7=15.$$

(Children understand this equation as a word problem: When 7 is added to the unknown (hidden) number x, 15 is obtained. Find x.). When they have found that the hidden number was 8, they write: $x=8$. To help them to acquire easier how the checking runs, a box as a place holder can be used. For instance,

$$\begin{array}{l} x+7=15\\ 8\\ \downarrow\\ \boxed{}+7=\boxed{8}+7=15\\ \downarrow\\ x \end{array}$$

Such a scheme suggests that x falls out of its place and 8 takes that place. Let us remark that such tiny procedures are valuable parts of didactical elaboration and they should never be ignored.

Another popular form of evaluation of literal expressions is found in many textbooks (particularly often in Russian ones), when children are assigned to fill in tables as, for instance, this one is

m	3	5	9	15	24	57
$27-m$						

An explanation in words of what is required may suffice, but it is still better to help children see clearly what they are supposed to do.

For instance,

(i) $m=3$

$$\begin{array}{l} 27-m \\ \quad 3 \\ \quad\downarrow \\ 27-\boxed{} = 27-\boxed{3}=24 \\ \quad\downarrow \\ \quad m \end{array}$$

(ii) $m=5$

$$\begin{array}{l} 27-m \\ \quad 5 \\ \quad\downarrow \\ 27-\boxed{} = 27-\boxed{5}=22 \\ \quad\downarrow \\ \quad m \end{array}$$

etc.

As soon as children learn how a variable is substituted by its specific values, such technicalities are omitted.

The third graders are supposed to assimilate the meaning of the term "takes value", which is particularly convenient when variation of literal expressions is concerned. It can be done by a parallel description of the cases as in (i) and (ii) above.

(i) When $m=3$, the expression $27-m$ takes the value 24.

(ii) When $m=5$, then $27-m=22$.

When m takes the value 5, the expression $27-m$ takes the value 22. Etc.

6. Using set-theoretical notations for the first time

The process of forming ideas of natural numbers begins with activities of observation of collections of concrete objects. Using the examples of collections of objects that are meaningfully organized as coherent wholes, the terms "set" and "element" are to be used and, in that way, children assimilate gradually their meaning. When the idea of set is related to collections of concrete objects, then we say that such a concept is at the sensory level. (See [9], where a way of treating sets at that level is suggested).

As the next level, we take the case when examples are still finite sets but when the elements are conventional signs—symbols for numbers and letters. It is important to say that the shape of such signs serves only for their clear recognition and, laying stress on equating, also to say that such two signs are equal whenever they mean the same (represent the same number, letter, etc.).

Now we sketch shortly how the symbols: $\{\dots\}$, $\in$, $\notin$ are introduced to and used by children.

(i) Write all numbers belonging to the first ten: ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, (1, 2, 3, 4, 5, 6, 7, 8, 9, 10). Taken together, these numbers form a set which is denoted by

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

(ii) The letters: a, e, i, o, u are vowels. Taken together, these letters form a set which is denoted by

$$\{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}.$$

(iii) $\{32, 34, 36, 38, 40\}$ is the set of all even numbers belonging to the ____________ (fourth ten).

(iv) The set $\{1, 24, 125\}$ has ____ (3) elements. Its elements are: ____, ______, _________ (1, 24, 125). Etc.

(v) When a set is denoted by a letter, then that letter is in the upper case. When we use "A" to denote the set $\{2, 4, 6, 8, 10\}$, then we write

$$A = \{2, 4, 6, 8, 10\}.$$

Denoting in the same way another set, say, $\{1, 3, 5, 7, 9\}$ we use another letter, say, "B" and then we write

$$B = \{1, 3, 5, 7, 9\}.$$

The elements belonging to the set A are: ____, ____, ____, ____, ______ (2, 4, 6, 8, 10). To write shortly that 2 is an element of the set A, we use the sign "$\in$" (which is read "is an element of"). Thus, we can write

$$2 \in A, \quad \underline{\quad\;} \,(4) \in A, \quad \underline{\quad\;} \,(6) \in A, \quad \underline{\quad\;} \,(8) \in A \quad \text{and} \ \underline{\quad\;} \,(10) \in A.$$

The number 1 is not an element of the set A. Then, we write

$$1 \notin A$$

(and the sign "$\notin$" is read "is not an element of").

Write one of the signs "$\in$", "$\notin$" so that you obtain true relations:

$$4 \,\underline{\quad\;}\, (\in)\ A, \quad 5 \,\underline{\quad\;}\, (\notin)\ A, \quad 7 \,\underline{\quad\;}\, (\notin)\ A, \quad 10 \,\underline{\quad\;}\, (\in)\ A$$
$$2 \,\underline{\quad\;}\, (\notin)\ B, \quad 3 \,\underline{\quad\;}\, (\in)\ B, \quad 7 \,\underline{\quad\;}\, (\in)\ B, \quad 10 \,\underline{\quad\;}\, (\notin)\ B$$

Read these relations, checking if all of them are true (or you have made some mistakes).

Write A or B so that you obtain true relations:

$$6 \in \_\_\_\_ (A), \quad 7 \in \_\_\_\_ (B), \quad 6 \notin \_\_\_\_ (B), \quad 7 \notin \_\_\_\_ (A).$$

Read these relations that you have written, checking if all of them are true.

Using a number of similar examples children learn the meaning of and the way how the symbols: $\{\dots\}$, $\in$, $\notin$ are used.

To be able to represent those sets that are solutions of inequalities, children have to learn how to denote the sets of numbers having a greater number of elements (say, hundreds of them). But it is connected with the equating of sets and our next exercises are concerned with comparison of two sets and where children are supposed to find if they are equal or different. Let us recall that a set is equal only to itself and that the equating of sets is, in fact, equating of their different notations.

(vi) One and the same set can be denoted in different ways writing its elements in different order.

(a) Notations

$$\{4,2,1,3\}, \quad \{3,1,2,4\}, \quad \{4,1,3,2\}$$

represent one and the same set, whose elements are: 1, 2, 3, 4.The neatest of all such notations is the following one:

$$\{1,2,3,4\},$$

when these numbers are written in *increasing order*.

(b) Notations:

$$\{4,7,5\}, \quad \{7,5,4\}, \quad \{7,4,5\}$$

represent one and the same set, whose elements are: ___, ___, ___ (4, 5, 7). The neatest of all such notations is the following one:

$$\{4,5,7\},$$

when the numbers are written in ________________ order.

(c) Notations:

$$\{e,c,b,a,d\}, \quad \{a,e,d,b,c\}, \quad \{d,c,a,e,b\}$$

represent one and the same set, whose elements are: ___, ___, ___, ___, ___, (a, b, c, d, e). The neatest of all such notations is the following one:

$$\{a,b,c,d,e\},$$

when the letters are written in *alphabetical order*.

(vii) As you know it, when the elements of a set are numbers, the neatest notation is that when the numbers are written in increasing order.

For instance, the neatest notations are the following ones:

(a) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,

(b) $\{21, 22, 23, 24, 25, 26, 27\}$,

(c) $\{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$,

(d) $\{31, 33, 35, 37, 39\}$.

Not writing all numbers, these notations can be shortened as:

(a) $\{1, 2, \ldots, 12\}$ – denoting so the set of all numbers from 1 to 12,

(b) $\{21, 22, \ldots, 27\}$ – denoting so the set of all numbers from 21 to 27,

(c) $\{10, 20, \ldots, 100\}$ – denoting so the set of all tens belonging to the first hundred,

(d) $\{31, 33, \ldots, 39\}$ – denoting so the set of all odd numbers belonging to the fourth ten.

Denote in the short way:

(e) The set of all numbers belonging to the seventh ten:

$$\{\underline{\quad}, \underline{\quad}, \ldots, \underline{\quad}\} \qquad (\{61, 62, \ldots, 70\}).$$

(f) The set of all even numbers belonging to the ninth ten:

$$\{\underline{\quad}, \underline{\quad}, \ldots, \underline{\quad}\} \qquad (\{82, 84, \ldots, 90\}).$$

(g) The set of all numbers devisable by 5 which are less than 41:

$$\{\underline{\quad}, \underline{\quad}, \ldots, \underline{\quad}\} \qquad (\{5, 10, \ldots, 40\}).$$

Etc.

7. The simplest inequalities

In the same way as how a literal expression represents a whole set of concrete numerical expressions, an inequality containing a variable (an unknown) stands for a whole set of concrete numerical inequalities. Thus, the first examples designed to establish the meaning of inequalities $x < a$ and $x > a$, have to express this idea.

(i) When x takes the value:	the inequality $x < 4$ becomes:	being:
0	$0 < 4$	T (true)
1	$1 < 4$	T
2	$2 < 4$	T
3	$3 < 4$	T
4	$4 < 4$	F (false)
5	$5 < 4$	F
6	$6 < 4$	F
...	...	...

The values of x for which the inequality $x < 4$ is true are: ____, ____, ____, ____ (0, 1, 2, 3) and those for which it is false are: ____, ____, ... , ____ (4, 5, ... , 1000).

(A third grader is acquainted with the block of numbers up to 1000).

The set of values of x for which the inequality $x < 4$ is true is $\{0, 1, 2, 3\}$ and the set of those for which it is false is $\{4, 5, \ldots, 1000\}$.

(ii) When x takes the value:	the inequality $x > 5$ becomes:	being:
0	$0 > 5$	F
1	$1 > 5$	F
2	$2 > 5$	F
3	$3 > 5$	F
4	$4 > 5$	F
5	$5 > 5$	F
6	$6 > 5$	T
7	$7 > 5$	T
...	...	...

The values of x for which the inequality $x > 5$ is true are: ____, ____, ... , ________ (6, 7, ... , 1000) and those for which it is false are: ____, ____, ... , ____ (0, 1, ... , 5). The set of values of x for which the inequality $x > 5$ is true is { ____, ____, ... , ________ } ($\{6, 7, \ldots, 1000\}$) and the set of those for which it is false is { ____, ____, ... , ____ } ($\{0, 1, \ldots, 5\}$).

The values of x for which an inequality is true are called *its solutions* and the set of such values is called the *set of its solutions*.

(i') The solutions of the inequality $x < 4$ are the numbers: ____, ____, ____, ____ (0, 1, 2, 3) and the set of its solutions is {____, ____, ____, ____} ($\{0, 1, 2, 3\}$).

(ii') The solutions of the inequality $x > 5$ are the numbers: ____, ____, ... , _______ (6, 7, ... , 1000) and the set of its solutions is $\{$____, ____, ..., _______$\}$ ($\{6, 7, \ldots, 1000\}$).

Remember that *to solve an inequality* will mean to find the set of its solutions.

(iii) Write the inequality whose set of solutions is:

(a) $\{0, 1, \ldots, 100\}$ ________________ ($x < 101$)

(b) $\{100, 101, \ldots, 1000\}$ ________________ ($x > 99$)

(c) $\{101, 102, \ldots, 1000\}$ ________________ ($x > 100$). Etc.

8. Solving inequalities

Since the way of solving inequalities that we will present is based on properties expressing the variation of sums and differences depending on the change of value of their components, three different cases will be considered separately. Learning arithmetic, children do exercises expressing procedurally these properties as, for example, in the cases that follow.

(i) Put "<" or ">" so as to get a true inequality:

(a) $5 + 7$ ____ $5 + 11$, $5 + 11$ ____ $3 + 11$, etc.

(b) $17 - 4$ ____ $15 - 4$, $17 - 4$ ____ $19 - 4$, etc.

(c) $17 - 6$ ____ $17 - 8$, $17 - 6$ ____ $17 - 4$, etc.

(For the first graders)

(ii) Let the difference $517 - 389$ be given.

(a) When the minuend is increased (decreased) by 125, the given difference is increased (decreased) by ________ (125).

(b) When the subtrahend is increased (decreased) by 149, the given difference is decreased (increased) by ________ (149). Etc.

(For the third graders)

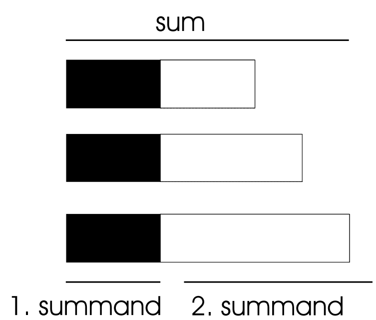
These properties can be clearly interpreted in an iconic way and children also learn to express them rhetorically. Now children have to learn how to express them depending on the change of a variable component denoted by a letter, say, x. Of course, a number of exercises which help children to understand what kind of a requirement are those somewhat less simple inequalities, should again be given. For instance:

When x takes the value:	the inequality $7 - x > 4$ becomes:	being:
0	$7 - 0 > 4$	T
1	$7 - 1 > 4$	T
2	$7 - 2 > 4$	T
3	$7 - 3 > 4$	F
4	$7 - 4 < 4$	F
...	...	...

Also such exercises can be used to explain solving of inequalities as the search for those values of x for which an inequality is true.

Proceeding further we will sketch a didactical elaboration showing how each of the three types of inequalities is solved.

8.1. The case of variable summand



You already know that a sum becomes larger—*increases*, when one of its summands becomes larger—*increases*. You also know that a sum becomes smaller—*decreases*, when one of its summands becomes smaller—*decreases*.

Let us denote by x that variable summand. Then a sum, for example, $176 + x$ changes its value depending on the change of value of the summand x. Thus, when

the value of x is:	the value of $176 + x$ is:
853	$176 +$ ________ $=$ ________,
1214	$176 +$ ________ $=$ ________,
5823	$176 +$ ________ $=$ ________.

8.1.1. Which value does x take, when the sum $1421 + x$ takes the value:

(a) 2633?	(b) 5888?	(c) 7989?
$1421 + x = 2633$	________________	________________
$x = 2633 - 1421$	________________	________________
$x =$ ____________	$x =$ ____________	$x =$ ____________

(Without forcing the use of the term "value", a simpler formulation of these exercises would be: Find x, when $1421 + x$ equals 2633, etc.)

8.1.2. Write the inequality which expresses the following:

(a) The value of the sum $725 + x$ is smaller than 1856: ____________________ ($725 + x < 1856$).

(b) The value of the sum $725+x$ is larger than 1856: ____________________ ($725 + x > 1856$).

To solve the inequalities

(a) $725 + x < 1856$ (b) $725 + x > 1856$,

first you solve the equation:

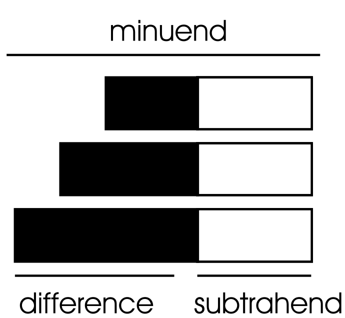
$$725 + x = 1856$$
$$x = 1856 - 725$$
$$x = 1131.$$

The sum $725 + x$ takes the value 1856, when x takes the value 1131. Hence, the value of the sum $725 + x$ will be

(a) smaller than 1856, when
$$x < 1131$$
and the set of solutions of the inequality
$$725 + x < 1856$$
is
$$\{0, 1, 2, \dots, 1130\}$$

(b) larger than 1856, when
$$x > 1131$$
and the set of solutions of the inequality
$$725 + x > 1856$$
is
$$\{1132, 1133, \dots\}.$$ Etc.

8.2. The case of variable minuend



You already know that a difference becomes larger—increases, when its minuend becomes larger—increases. You also know that a difference becomes smaller—decreases, when its minuend becomes smaller—decreases.

Let us denote by x that variable minuend. Then, a difference, for example $x - 1111$ changes its value depending on the change of value of its minuend x. Thus, when

the value of x is:	the value of $x - 1111$ is:
5831	________ $- 1111 =$ ________
8225	________ $- 1111 =$ ________
17141	________ $- 1111 =$ ________

8.2.1. Which value does x take, when the difference $x - 1111$ takes the value:

(a) 33?	(b) 854?	(c) 24801?
$x - 1111 = 33$	____________	____________
$x = 1111 + 33$	____________	____________
$x =$ ________	$x =$ ________	$x =$ ________

8.2.2. Write the inequality which expresses the following:

(a) The value of the difference $x-2315$ is smaller than 72: ____________ ($x - 2315 < 72$),

(b) The value of the difference $x - 2315$ is larger than 72: ____________ ($x - 2315 > 72$).

Now we will see how the inequalities:

(a) $x - 2315 < 72$ (b) $x - 2315 > 72$

are solved. For example, when x takes the value 1000, the difference $x - 2315$ becomes $1000 - 2315$ and such a subtraction is not feasible. And as you know it, the subtraction is not feasible whenever minuend is smaller than subtrahend. Therefore, all solutions of these two inequalities must belong to the set of numbers

$$D = \{2315, 2316, \dots\}.$$

Here we also solve first the equation:

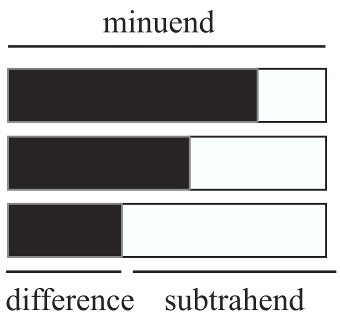
$$x - 2315 = 72$$
$$x = 2315 + 72$$
$$x = 2387.$$

Thus, we see that the difference $x-2315$ takes the value 72, when x takes the value 2387. Then, the difference $x - 2315$ will be

(a) smaller than 72, when
$x < 2387$
and all solutions of the inequality
$x - 2315 < 72$
belong to the set D.
Hence, the set of its solutions is
$\{2315, 2316, \dots, 2386\}$

(b) larger than 72, when
$x > 2387$
and all solutions of the inequality
$x - 2315 > 72$
belong to the set D.
Hence, the set of its solutions is
$\{2388, 2389, \dots\}$.

8.3. The case of variable subtrahend



You already know that a difference becomes larger—increases, when its subtrahend becomes smaller—decreases. You also know that a difference becomes smaller—decreases, when its subtrahend becomes larger—increases.

Let us denote by x that variable subtrahend. Then the value of a difference, for example, $876 - x$ changes when the value of x changes. Thus, when

the value of x is:	the value of $876 - x$ is:
50	$876 - \underline{\qquad\qquad} = \underline{\qquad\qquad}$,
200	$876 - \underline{\qquad\qquad} = \underline{\qquad\qquad}$,
500	$876 - \underline{\qquad\qquad} = \underline{\qquad\qquad}$.

8.3.1. Which value does x take when the difference $9987 - x$ takes the value:

(a) 33?	(b) 900?	(c) 9000?
$9987 - x = 33$	$\underline{\qquad\qquad\qquad}$	$\underline{\qquad\qquad\qquad}$
$x = 9987 - 33$	$\underline{\qquad\qquad\qquad}$	$\underline{\qquad\qquad\qquad}$
$x = \underline{\qquad\qquad}$	$x = \underline{\qquad\qquad}$	$x = \underline{\qquad\qquad}$

8.3.2. Write the inequality which expresses the following:

(a) The value of the difference $1876-x$ is smaller than 200: $\underline{\qquad\qquad\qquad}$ $(1876 - x < 200)$,

(b) The value of the difference $1876 - x$ is larger than 200: $\underline{\qquad\qquad\qquad}$ $(1876 - x > 200)$.

Now we will see how the inequalities:

(a) $1876 - x < 200$ (b) $1876 - x > 200$

are solved.

The subtraction $1876 - x$ is feasible only when x takes values which are not larger than 1876. Thus, all solutions of these inequalities belong to the set

$$D = \{0, 1, 2, \ldots, 1876\}.$$

First we solve the equation:

$$1876 - x = 200$$
$$x = 1876 - 200$$
$$x = 1676.$$

We see that the difference $1876 - x$ takes the value 200, when x takes the value 1676. Then, the value of the difference $1876 - x$ will be

(a) smaller than 200, when
$$x > 1676$$
and the solutions of the inequality
$$1876 - x < 200$$
belong all to the set D.
Hence, the set of its solutions is
$$\{1677, 1678, \ldots, 1876\}.$$
Etc.

(b) larger than 200, when
$$x < 1676$$
and the solutions of the inequality
$$1876 - x > 200$$
belong all to the set D.
Hence, the set of its solutions is
$$\{0, 1, 2, \ldots, 1675\}.$$

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