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COMPARING TWO WAYS OF ELABORATION OF COMPLEX NUMBERS

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Abstract. The aim of this article is to compare two well-known ways of introducing complex numbers. As a result, we find that the treatment of complex numbers as polynomials in one variable i, is the most acceptable for students. The concepts of ring and field play a hidden but crucial role in such an approach.

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Introduction

It is a long tradition at Bulgarian technical universities to introduce complex numbers as ordered pairs of real numbers, in the first year courses of mathematics [1]. In contact with these students, one can easily witness quite a lot of embarrassment caused by such an approach. An analogy with vectors do exist when addition and multiplication by real numbers is concerned. But this analogy ends with the multiplication formula

$$(a;b)(c;d) = (ac - bd; ad + bc),$$

which occurs to be quite strange for these students. In addition, identification of the pair $(a, 0)$ with the real number a and taking that $(0, 1)$ is a kind of special unit come as two unusual conventions. And at the end, the admission that $a + bi$ is a more convenient notation than (a, b) still comes.

The author's experience is that such formal approach causes some psychological barriers found even in the way when students solve quadratic equations having negative discriminates. Avoiding such a superficial formalism, we sketch here a more natural approach.

1. An outline of our approach

It is widely recognized that in the courses for students not studying pure mathematics, the number of new concepts should be as little as possible. The effectiveness of the teaching process also increases if already learnt concepts and skills are combined with the new topics subjected to elaboration.

Our idea is that complex numbers are best elaborated when treated as polynomials in one variable factorized by equality. This is, of course, a procedure which has its ground in Bulgarian (and many other) secondary school curricula. Our evident inspiration is Kleinian idea of looking at the elementary mathematical topics from the higher mathematics view point.

2. Rings and fields in secondary school programs

According to the Bulgarian secondary school curriculum, the integers are scheduled to be learnt by sixth graders. This is the earlier stage when students meet an algebraic ring in an implicit form. In the course of that grade they are supposed to learn these new object and their main properties as well as to develop operative skills.

The fifth graders learn fractions identifying notations by reducing and enlarging them according to:

$$\frac{p}{q} \equiv \frac{p'}{q'} \iff pq' = p'q, \quad p, q, p', q' \in \mathbf{N}.$$

On the basis of such an equivalence, the sixth graders cover the whole field of rational numbers

$$\mathbf{Q} = \left\{ \frac{p}{q} : \ p \in \mathbf{Z}, \ q \in \mathbf{N} \right\} / \equiv .$$

The seventh grade curriculum includes polynomials with rational coefficients

$$a + bx + \cdots + cx^n, \quad a, b, \ldots, c \in \mathbf{Q}, \ n \in \mathbf{N}$$

and it is expected that the students gain a sufficient skill in adding, subtracting, multiplying and expanding them. This time, they meet again an example of algebraic ring over a field, implicitly.

Continuing to deal with polynomials, the eighth graders learn the fact that the quotient of two polynomials generally is not a polynomial. Starting with the square root of 2, the ninth graders begin to be acquainted with irrational numbers. Particularly, in the form of an exercise assigned to them, they prove that sum, difference, product and quotient of the two numbers of the form

$$a + b\sqrt{2}, \ a, b \in \mathbf{Q}$$

is again a number of the same form. In fact, they prove that the set

$$\mathbf{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}$$

is a field, being an extension of the field of rational numbers. A particular attention is paid to the division done as rationalization of denominators via multiplication by the conjugate expressions

$$\frac{a + b\sqrt{2}}{c + d\sqrt{2}} = \frac{(a + b\sqrt{2})(c - d\sqrt{2})}{(c + d\sqrt{2})(c - d\sqrt{2})} = \frac{1}{c^2 - 2d^2}(a + b\sqrt{2})(c - d\sqrt{2}).$$

The tenth grade curriculum includes the main facts related to polynomials over $\mathbf{R}$, of higher degree than two.

3. Factorizing $\mathbf{R}[i]$

As it was mentioned above, the ring

$$\mathbf{R}[x] = \{a + bx + \cdots + cx^n : a, b, \ldots, c \in \mathbf{R}, \ n \in \mathbf{N}\}$$

is thoroughly studied throughout the grades 7–10, considering the set of real numbers $\mathbf{R}$ as its subset. Let us add also that the complex numbers are an optional theme in Bulgarian twelfth grade curriculum.

Therefore, we see that the first year university students are one step close to complex numbers and that they have a proper preparation to learn them. (Polynomials in another than x variable, the students meet, for instance, in physics studying movements with acceleration: $s(t) = s_0 + vt + at^2$.) Thus, the complex numbers can be given as a ring

$$\mathbf{R}[i] = \{a + bi + \cdots + ci^n : a, b, \ldots, c \in \mathbf{R},\ n \in \mathbf{N}\},$$

cut by the equality

$$i^2 \equiv -1,$$

which is used to reduce polynomials of degree higher than 1, to the linear ones. In other words, students construct

$$\mathbf{C} = \mathbf{R}[i]/(i^2 \equiv -1) = \{a + bi : a, b \in \mathbf{R}\}.^1$$

The process of factorization runs by expanding products of two polynomials in the usual way and then by applying the factorization equality. In such a two-step procedure, otherwise a 'monster' formula, comes quite naturally

$$(a+bi)(c+di) = ac+(ad+bc)i+bdi^2 = ac+(ad+bc)i-bd = (ac-bd)+(ad+bc)i.$$

In order to prove that $\mathbf{C}$ is a field, it remains to show that the quotient of two linear polynomials in i is again a polynomial of the same kind. But it is easy for students and then, they multiply a quotient of two complex numbers by the conjugate of denominator

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{1}{c^2-d^2i^2}(a+bi)(c-di) = \frac{1}{c^2+d^2}(a+bi)(c-di).$$

4. Conclusions

As it follows from our forerunning exposition, complex numbers can be elaborated as something almost deja vu. By using the well-known mathematical concepts and the already established skills, this new kind of numbers is better accepted by students.

Of course, the idea of presenting complex numbers in the form $a + bi$ is as old as the complex numbers themselves. But in a process of teaching even small details could play a crucial role in understanding a mathematical content. This author's teaching experience is that a look at mathematical structures may highlight the way of elaboration of complex numbers.

- It is not necessary to formulate as new rules properties of the basic algebraic operations when students already know them as rules of performing.

[1] Indeed the set of multiples of $1 + i^2$ is a main ideal in $\mathbf{R}[i]$

- Preserving all main operative properties and accepting spontaneously the inclusion $\mathbf{R} \subset \mathbf{C}$, students are apt to acquire complex numbers as really be a specific kind of numbers.
- Using of the formula for roots of quadratic equations stays the same as it was the case when an irrational root occurs.

Concluding, let us say that the already mentioned Klein's idea could still guide us in teaching courses for the university students.

REFERENCES

1. Kassabov, O., Nikolov, K., *Higher Mathematics*, HST, Sofia, 2003. (in Bulgarian)

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