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A BROADER WAY THROUGH THEMAS OF ELEMENTARY SCHOOL MATHEMATICS, V

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Abstract. Decimal notation and decade grouping are the basis of traditional elaboration of arithmetic. In the approach that we follow, this elaboration goes gradually through the didactical structuring and extending of number blocks. In this process of block extending, we use sums, applying the principle of permanence of the meaning of addition. Then, finding decimal notations for such sums is a purely calculation task, that is reducible to easier and smaller (already performed) cases.

In this paper, the block 1-100 is constructed as the extension of the block 1-20 and its didactical analysis is executed. Due to pedagogical demands, we suggest the replacement of some arithmetic rules by their "narrative" substitutes that also have to be meaning-based.

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10. A further block extension

When we say "children already know", it means that they know how to use the (fragments of) concepts they have learnt and how to act upon the assigned demands within the frame of activities they have gone through. Based on such an amount of knowledge, a further building of the acted-out arithmetic progresses. But this process of building has to be planed upon a clear structural comprehension of the content. Here we start to treat the number block up to 100, conducting didactical analysis of that piece of arithmetic matter. Though our exposition is very concrete, it is, in no way, a didactical transposition of that theme. In fact, we will be structuring the content to be a road along which the learning activities should develop.

10.1. Signs and language. In this process of block building, two kinds of iconic signs are used—pictographs which are simple drawings, representing real world situations in the way that displays more clearly their mathematical structure and ideographs which are drawings suggesting the meaning. As conveyors of the meaning, the latter signs have to be chosen and used with a great care. For us, the ideographs may serve for an outer, visual expression of our imagery, but for children, they are paradigmatic signs, that serve them to grasp the meaning of concepts in the state of their creation.

Number blocks are examples par excellence of the Vigotsky's systems of scientific concepts. A block is a set of numbers, each one of them being an individual concept and all of them being interrelated by means of arithmetic operations and relations. Thus, at the start, mathematical concepts are fragments of scientific concepts, and to quote Vigotsky again, whose full meaning will be developed only after years of learning. It is often seen in textbooks and some "research" papers that the spontaneous meaning of some basic mathematical concepts is confused with their sharp scientific meaning which starts to be developed at the very beginning of school mathematics. That is why the requirement for the precision of mathematical language is an essential didactical task.

The Bruner's idea of narrative forms, exposed in his book [16], could be too enticing if not taken with a reasonable limitation. Keeping our attention inside the limits of the part of arithmetic content which is under consideration here, we take that the language used to describe the real world situations and to discover their underling mathematical structure, has to be picturesque and vivid. The same holds true for the attempts to inject life to pictographs when they have to be made closer to the real situations they represent. Weaving into a situation a story for the purpose of pedagogical dramatization (producing a cognitive conflict) also is an excellent example of narration with fine effects to its mathematical resolution. Describing our activities, we often use narrative forms as, for instance, "building of number blocks", "seeing at once", etc. The reader, interested in the fine aspects of such a use of the language, is directed to the refined exposition of narration found in the above mentioned Bruner's book.

Since the time of the historic Pestalozzi's school, the ultimate aim of arithmetic teaching has been creation of abstract concepts. And the modus operandi of achieving this aim is the use of concepts abstractly, that is to say to use them independently of any their specific meaning. And when a specific meaning is attached, operating with arithmetic codes proceeds separately, outside of the imaginary line framing the given meaning. Whenever we are on the abstraction side of this line, the language we use is the subject vocabulary of mathematics. Not counting grammatical modifications, this way of expression is fixed, all terms are taken with their literal meaning (and no other exists within this kind of activities). A teacher threads along the way to abstraction, whenever he or she uses the subject vocabulary, having in mind concepts, not their examples or iconic representations.

Now I bring to my mind a television programme in which, a psychologist talked to a boy, a second grader, asking him questions from mathematics in order to estimate the degree of his understanding. "How much is it when 28 is divided by 4", was one of her (the psychologist's) questions. "Seven", the boy answered. "Can you explain why it is seven", she went on. And while she was mumbling her senseless hint "28 marbles, 4 boys, ... " the right explanation came out from the boy's mouth, "because 7 times 4 is 28".

This story demonstrates very well why the acquaintance with the didactical structure of the subject matter is so important for a teacher as well as it shows how a metacognitive attempt without such a knowledge may be below the cognition that it is concerned with. And when arithmetic facts are dipped again and again into the meaning they have sprung from, it certainly means the slowing of the abstract development of the child's thought.

10.2. Principle of permanence of the meaning of operations. After covering the number block up to 20, it is supposed that children have learnt numbers and operations in this block Thus, for them, for example, the sums as

$$15 + 10, \qquad 10 + 10 + 10$$

are already meaningful. Stimulated by means of place holders (or in some other way), they know how to transform them into, say,

$$10 + 15, \qquad 20 + 10$$

respectively, but they need not know yet for the decimal notations: 25, 30 of these numbers. As a matter of fact, it should be instructive to do a number of similar exercises before this block extension begins.

Extending the block 1-20 on the additive base, the first sums to be named and denoted in decimal system will be those of tens. Thus,

$20 + 10$ will be denoted by 30 and read: thirty,

$30 + 10$ will be denoted by 40 and read: forty,

...

$90 + 10$ will be denoted by 100 and read: hundred.

Writing

$$20 + 10 = 30, \quad 30 + 10 = 40, \quad \ldots, \quad 90 + 10 = 100,$$

two kinds of notation are equated, one already known to children and the other one new to them. The sense of this equating will again be the same: different symbols denoting the same number are equated. Now simple calculations with tens: $40 + 30 = 70$, $9050 = 40$, etc. follow this first step.

The next to do is the introduction of decimal notations for sums of tens and units. Thus, for example,

$20 + 7$ will be denoted by 27 and read: twenty-seven,

$50 + 4$ will be denoted by 54 and read: fifty-four,

$90 + 1$ will be denoted by 91 and read: ninety-one,

and so on, expecting here a spontaneous induction from the side of children. Equalities as:

$$20 + 7 = 27, \quad 50 + 4 = 54, \quad 90 + 1 = 91, \quad \text{etc.}$$

have the meaning as already said.

As a parallel with sums, the equalities comprising differences, as the following ones

$$27 - 7 = 20, \quad 27 - 20 = 7, \quad 54 - 4 = 50, \quad 54 - 50 = 4, \quad \ldots$$

have their right turn to be included (stressing again the interrelation between the two operations).

After this illustration by examples how a block extension is carried out, now we state the principle of such extensions: *once established meaning of addition (and* *subtraction) is taken to be permanent.* This is to say, that meaning is the same for all numbers figuring as the components of sums (and differences) and the already established properties of these operations remain to hold. Together with addition, the subtraction is (at least implicitly) combined and that is why we do not separate these operations in this context.

Thus, with the extension of the set of numbers $\mathbf{N}_{20}$ to the set of numbers up to 100, denoted by $\mathbf{N}_{100}$, and by the preservation of meaning of the already introduced operations and relations, a larger structure

$$\{\mathbf{N}_{100}, +, -, =, <\}$$

is formed and called the block of numbers up to 100.

In this structure, a sum or a difference as, for example,

$$54 + 37, \qquad 62 - 28$$

has a full meaning and what we have to cover is the way how such expressions are transformed into 91 and 34, respectively. This way of transforming is clearly a calculation task.

Let us add also that counting is an ordered way of reciting names of numbers. For children to learn to count up to 100 (in one, two, five, ...) is certainly something what should not be neglected. For comparison of numbers, we also suggest the way we illustrate by an example:

Filling in, write one of the signs "=", "<", so that all equalities and inequalities are true:

$$36 \_\_ 30 + 6 \_\_ 30 + 10 \_\_ 40 \_\_ 40 + 10 \_\_ 50 \_\_ 50 + 2 \_\_ 52$$

where the transitivity of these relations is actively used as well as the property of addition: "bigger the summand, bigger the sum". Children will, of course, easily compare these numbers on the hearing of their names, but to do a number of comparisons, as the one shown, is still instructive.

10.3. Calculation cases. Using specific examples of addition and subtraction in the order of their complexity, we shall illustrate here how, step by step, such a calculation task is accomplished in this block.

$$\begin{aligned}
\mathbf{4}0 + \mathbf{2}5 &= (\mathbf{4}0 + \mathbf{2}0) + 5 = \mathbf{6}0 + 5 = \mathbf{6}5\\
\mathbf{5}2 + 6 &= \mathbf{5}0 + (2 + 6) = \mathbf{5}0 + 8 = \mathbf{5}8\\
\mathbf{2}5 + 7 &= \mathbf{2}0 + (5 + 7) = \mathbf{2}0 + \mathbf{1}2 = \mathbf{3}2\\
\mathbf{2}6 + \mathbf{3}8 &= (\mathbf{2}0 + \mathbf{3}0) + (6 + 8) = \mathbf{5}0 + \mathbf{1}4 = \mathbf{6}4.
\end{aligned}$$

In the first steps of performing these operations, coloured digits should be used (say, the blue ones to denote tens and the red to denote units). The way how these calculations are carried out can also be formulated as a "narrative rule": *units are added to units and tens to tens.*

Cases of subtraction go as follows,

$$46 - \mathbf{20} = (40 - \mathbf{20}) + 6 = 20 + 6 = 26$$
$$38 - \mathbf{6} = 30 + (8 - \mathbf{6}) = 30 + 2 = 32$$
$$47 - \mathbf{24} = (40 - \mathbf{20}) + (7 - \mathbf{4}) = 20 + 3 = 23$$
$$43 - \mathbf{8} = 30 + (13 - \mathbf{8}) = 30 + 5 = 35$$
$$61 - \mathbf{27} = (50 - \mathbf{20}) + (11 - \mathbf{7}) = 30 + 4 = 34.$$

Here we have coloured subtrahends and their parts. The "narrative rule" of subtraction is: *units are subtracted from units and tens from tens.*

In both cases of addition and subtraction, the sketched methods reduce the calculation cases to: calculation with tens and that in the block up to 20. This is also the right place when vertical addition and subtraction should start. The more complex cases are those when carrying and borrowing of one ten appears.

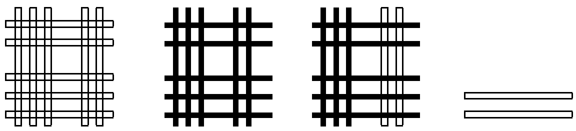
10.3.1. Actual didactical transpositions require narrative forms. Looking in a purely structural way (regardless of any pedagogical considerations), foregoing calculations are based on arithmetic rules which, in their symbolic form, are the following

$$(a + b) - c = (a - c) + b,$$
$$(a + b) - c = a + (b - c),$$
$$(a + b) - (c + d) = (a - c) + (b - d).$$

Meaning-based, these rules have their right place other than this, in the four year didactical elaboration of arithmetic (at least, in their procedural and rhetoric form), when their explicit formulation is an act of clear structuring of school arithmetic. Later, having been known to children, they may serve as a means of further extension of the system **N** of natural numbers (say, to the system **Z** of integers).

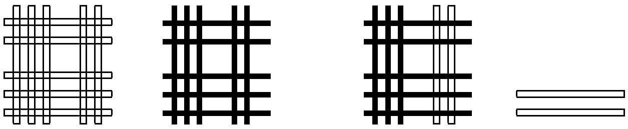
Realistically looking, these rules should not be given explicitly in an actual elaboration of the block 1–100. In explicit form, they would complicate such an elaboration and, when figuring in this context, they stand for teachers (and teachers of teachers) to understand fully the structure of this block. Instead, the use of the above two "narrative rules" fulfil the function of the general rules within this didactical unit. When saying that units are added to units and tens to tens, we do not express so a principal property of addition, but instead we describe how a procedure is carried out with numbers in their decimal notation. It does not mean yet that such procedural rules should be stated without also being meaning-based. Leaving out the complete set of details, we will demonstrate it, using just one more complex case: $32 - 18 = (20 - 10) + (12 - 8)$.

Looking at the following arrangement as a whole,



$32 - \mathbf{18}$

we see 32 sticks, 18 being red and $32 - 18$ blue. Looking at the same arrangement differently, we see the blue sticks in two groups,



$20 - \mathbf{10}$ $\qquad\qquad$ $12 - \mathbf{8}$

in one being $20 - \mathbf{10}$ blue sticks and in the other $12 - \mathbf{8}$. Altogether, it is $(20 - \mathbf{10}) + (12 - \mathbf{8})$ blue sticks. Differently reacting to the same set of blue sticks, we get two different notations of the same number which, when equated, produce the equality

$$32 - \mathbf{18} = (20 - \mathbf{10}) + (12 - \mathbf{8}),$$

which traces the right way of calculation, reducing a more complex case to two simple ones.

Let us note that in derivation of such rules, we see again the Cantor principle of invariance of number in action—looking at an arrangement of sticks in two different ways, two different expressions are obtained and ignoring the ways how we see these sticks grouped, the equality of these expressions follows.

10.3.2. Use of brackets. In terms of activities being carried out, the role of brackets is best expressible as a command: *first do what is in the brackets.* Only then, the use of these symbols goes with a full sense For example, the expressions:

a) $8 + (2 + 6)$, b) $17 - (11 - 6)$

have to be transformed in this way:

a) $8 + (2 + 6) = 8 + 8 = 16$,

b) $17 - (11 - 6) = 17 - 5 = 12$.

Then and only then, an equality expressing a rule as, for example, the following one

$$17 - (7 + 2) = (17 - 7) - 2$$

gains its right meaning: *we can calculate in either of the two ways commanded by brackets, obtaining so the same result.*

Longer sums can appear without any indicated way of summation, either as a result of "counting in groups" or as a result of leaving out the brackets on the basis of the rule of association of summands. In any case, such a notation does not cause even a slight embarrassment. Let us add also that this summation is usually accompanied with the teacher's direction: sum following the order of summands or in a way you find easiest.

A real embarrassment could appear with longer expressions combining the minus sign. For example, if

$$18 - 5 + 3$$

is given, some children will follow the order of the component numbers and calculate:

$$18 - 5 + 3 = 13 + 3 = 16,$$

while the other ones will be summing first and calculate:

$$18 - 5 + 3 = 18 - 8 = 10.$$

Both such groups of children could be right in their own way. With the brackets in two different position:

$$(18 - 5) + 3, \qquad 18 - (5 + 3)$$

two different results will, of course, follow.

At this level, before the stage of a more formal operating on numerical expressions (usually when the system **Z** of integers is covered), an important didactical demand is *to write and compose only realistic expressions, which have an interpretation by means of addition (subtraction) schemes.*

For example, Johnny is playing a game of wining and loosing marbles.

a) In the first turn he lost 5 marbles, having $18 - 5$ left. In the second turn he won 3 marbles. Now he has got $(18 - 5) + 3$ marbles.

b) In the first turn he lost 5 marbles and in the second 3. Altogether, he lost $5+3$ marbles, having left $18 - (5 + 3)$ marbles.

Hence, we see that both above expressions are realistic. The following syntactic rule regulates the use of brackets: *When longer expressions (longer sums and differences) are composed, brackets are placed around each component being itself a sum or a difference.*

Of course, the programmed exercises should be exploited to train children how to use brackets gradually. And how to use them correctly is a more acute didactical task than to think of their leaving out.

At the end of this subsection, we will tell again a story which stands as an outer part of this context.

A gentleman, a teacher of mathematics at a high school was lecturing. His audience were elementary school teachers and his lecture was on expressions combining more than one operation, taken to be a topic for third graders. When he was transforming the expression: $7 - 9 + 5$, in this way

$$7 - 9 + 5 = 5 + 7 - 9 = 12 - 9 = 3,$$

relying on the commutative law, a voice was heard asking: “Please, tell us which commutative law you are applying”. “Commutative”, was his short answer. “But there is no such a law at this level”, continued she (the teacher who spoke), asking him to compose a word problem leading to $7 - 9 + 5$. And while he was keeping his eyes closed in a vain attempt, the presiding person cut off this uncomfortable situation.

It was evident that the gentleman was applying the commutative law for integers. A sum of integers, as for example

$$7 + (-9) + 5$$

is often written as $7 - 9 + 5$, and for such sums the commutative law holds, but such a sum can never enter the arithmetic content for the third graders.

My personal thesis that good elementary school teachers should know the mathematical content that they teach, better and deeper than any mathematician not being professionally oriented to didactics, was at that moment in a state of confirmation.

10.4. Mental scheme-system-structure. It is pretty hard to fix the meaning of some general terms in an indented domain as the education is. In any case, to help understanding, it is desirable that the individual use of such terms is stated clearly.

Thus, we take a mental (cognitive) scheme to be a blend of mutually related mental images. To consider such a scheme as a whole, with all relating links is something far from being phenomenological (and equally far from the expertise of this author).

Following Vigotsky, we take the term "system" to mean a class of concepts together with all linking relations among them. When such a system is formulated in terms of sets and set theoretic relations, then we call it (mathematical) structure. Hence, a system of concepts and a structure (taken with the just described sense) are having nearly the same meaning, while the mental scheme is their reflection in our imagery.

Let us recall that we often hear (and say) that in the process of learning and abstracting, it is important to teach children discover mathematical structure of real world situations. This is to say that a high-noise appearance has to be replaced by a low-noise one which is, in the latter case, called mathematical model. Some simple examples are: two intersecting roads, when seen as two intersecting lines or a situation in the surrounding world comprehended as an addition scheme and represented iconically by a simple pictograph. In this and similar cases, the term "structure" cannot clearly be replaced by the term "system".

For a clear and extensive consideration of the concept of structure, the reader is directed to the chapter 7 of the Freudenthal's book [17].

At the end, let us say that we have not covered yet the complete structure of the block 1–100. When the structure

$$\{\mathbf{N}_{100}, +, -, =, <\}$$

is enriched with multiplication and division, this block shall be complete, what we will do in the next installment of this article.

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