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DIDACTICAL ANALYSIS—A PLAN FOR CONSIDERATION

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Abstract. Inspired by the idea of Freudenthal's didactical phenomenology, an integrated course of (didactics of) mathematics for primary school teachers is sketched. Based on history of mathematics and education, mathematics as science and psychology, didactical analysis of the subject matter is considered as the core of such a course. Only upon this analysis a proper shaping of didactical transposition of the subject matter is possible which would be, then, widely understood by teachers.

A particularly controversial theme—basics of mathematical logic in school is also discussed. A way of elaboration spanning the current contents of school mathematics is suggested, going by interpretation and without the use of truth tables.

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Issues for discussion:

- The role of history of mathematics in a course of teacher training.
- The integration of mathematics history, mathematical content and the didactics of teaching mathematics.
- Essential contents of courses of teacher training relating to geometry, number and logic.

1. Standing up for didactical analysis of the subject matter

Mathematics is often viewed as something very fixed and fossilized. From such a standpoint it follows that school teachers have to learn only one thing more—how to transfer it to their pupils. Arguing against, we start with the main points of this article.

All contents of the school mathematics have one of the following aspects:

- historical—when they are seen, as once they were, *in statu nascendi*,
- scientific—when they are seen exposed in a logically compact way,
- didactical—when they are transposed to be suitable for learning.

Related to the widely accepted principle that ontogenetic development follows phylogenetic one, some acquaintance of teachers with main facts from history of mathematics (and education) is indisputably important. A proper selection of these facts should always be dependent on the specific subject matter.

A good acquaintance with the scientific aspect serves a specialist in didactics of mathematics to do undertake a logical analysis of the relevant contents. To say it in other words, such knowledge is the ground upon which the meaning of the fundamental mathematical concepts is sharply established, which helps this specialist avoid formation of quasi-concepts and shifts in meaning (so often present in primary school textbooks). To some degree, such knowledge should be considered as useful even for primary school teachers.

If history of mathematics and mathematics as science are two pillars which support didactical analysis, the third is the science of learning and experience of teaching. Investigations of the ways we perceive and conceive were traditionally a preoccupation of philosophy. Nowadays, exposed to experimental verifications, it is predominantly the subject of psychology. Figuratively speaking, the psychologists now provide us with the landmarks which outline this area of interest, but they cannot tell us how to pave it. And this paving is the essence of didactical analysis.

Inspecting the textbooks from several countries is the best way to draw an outline of the state of affaires of mathematical education in its most relevant reality. Elementary school books are found to be full of blunders and, overall, a lot of serious misunderstanding is encountered. Based upon traditional courses in didactics of mathematics, many authors of these books show a complete lack of a deepened knowledge of the subject matter. Without any doubt, therefore, didactical analysis of the main teaching themes of school mathematics should be the core of all courses in mathematical didactics. This would then enable students at educational institutions to form a masterly knowledge of the subject matter so that when they are later in the classroom they would be able to follow the didactical transpositions of these themes with full understanding. Might it also encourage them to develop critical thinking instead of accepting everything they are told by a large variety of "experts" without questioning? If these observations indicate the grim actuality of educational practice, they also suggest teacher education as the first place of changes.

It is evident that our stand on didactical analysis is directly influenced by the views of Hans Freudenthal. Terms as "logical analysis of the subject matter", "didactical phenomenological analysis", "didactical phenomenology" indicate all those places in his books ([2], [3]), where the elements of didactical phenomenology of mathematical concepts are formed. Developed by Edmund Husserl, phenomenology insists on the intuitive foundation and verification of concepts, without regard to traditional epistemological questions.

As we suggest an approach to didactical analysis from "below", we will present here a selection of relevant topics and their uniting links, reflecting the internal integrity of the subject matter. In accordance with such a stand, we will avoid stating general conclusions and observations separately from their basis in concrete teaching material. These observations are rooted in the authors experience, based on courses he taught to students preparing to be primary school teachers and seminar work with them in which they were analysing and criticizing existing school books and then shaping pieces of didactical transposition, which they verified in their own school practice.

When in school, these students learnt Euclidean geometry which starts with undefined concepts and with postulated relations among them. This approach through scientific exposition should be contrasted with the geometric activities of recognizing, shaping and drawing that help a child gain basic understanding of such concepts. The existing practice of directing these students to take study further courses in mathematics is quite debatable—does not it seem better to let them learn again what they already know but in a deepened way?

2. Analysis of primary school geometry

If school geometry is basically a version of Euclidean geometry, no matter how simplified, then geometric ingredients in primary school programmes should be called 'pregeometry'. Thanks to J. Piaget and his experimental findings, we now know that a child spontaneously develops his/her own intuitive geometric ideas following the order: topological–projective–Euclidean. (For a mathematically established exposition of these ideas, see this authors paper in "The Teaching of Mathematics", vol. IV, 1, pp. 41-70; see also on Web: `http://www.komunikacija.org.yu/teachmat_e`). Certainly, these findings have provided an important impetus for selecting and arranging pregeometric topics.

2.1. Related history—a list of topics.

- A survey of activities of human beings from archaeological past: shaping of stone tools, decorations on ceramic objects, paintings on cave walls. Basic designs examples which peoples executed within their primitive civilizations. Shape as an inherent in real world objects. Geometry in prehistoric civilizations (in the East and Middle-East, Ancient Egypt).
- Birth of the first Grecian schools. Thales' proof that an angle inscribed in a semicircle is right and the beginning of logical thinking. Pythagoras and cognition of mathematical objects as being abstract ideas. Pythagorean comprehension of mathematics as the essence of the design of the universe. Erathostenes calculation of the length of the circumference of the Earth (contrasted with the apparent and fantastic Homerian vision of the world). Euclidean Elements and geometry conceived as a closed system containing causes of its own facts. Grecian conception of equal areas and volumes.

Comment. *Selected topics from history of geometry should expose the genesis of mathematical concepts relevant to primary school contents. Otherwise, without any search for their intuitive meaning, geometric facts and postulates, taken as a priori acceptable truths, lead into mystification and misunderstanding.*

2.2. Psychological preliminaries—a list of topics.

- Function of the eye (analogy with and difference from camera). Comprehension of perception. Principles of perception. Cues for seeing depth.
- Concepts as tripartite entities having for their constituents: the corresponding class of examples, the mental image and the name (including a possible symbolic sign). Bruners modes of enactive, iconic and symbolic representation. Iconic and symbolic representation of concepts. Geometric drawings as iconic signs.

- Comparison of concepts according to their degree of abstractness. The concept of set as the most general in regard to all other concepts of classical mathematics. Levels of abstractness. Concepts at sensory level.
- Definitions as sentences which determine a concept via another one of higher degree, plus *differentia specifica.* Premature tendencies to define or to prove.
- Ontogenetic development of the speech according to L. S. Vygotsky ([6]). Spontaneous and scientific concepts. Systems of concepts. Structures and cognitive schemes.

COMMENT. *It is assumed that the students have also taken one or more courses in psychology. This list selects those topics which are particularly useful for, and effectively interpretable with, the contents of primary school geometry and arithmetic.*

2.3. Itemization of geometric contents.

- Inherent geometry and the meaning of the words denoting place, positional relationships of the objects in the natural surroundings, directions of movement, etc.
- Perception of solid objects and formation of geometric ideas. Relationship "object–concept" and the reversibility of the childs thought.
- An intuitive description and discrimination of topological, projective and Euclidean properties. Conditioned and intentional ignoring of spatial extensions and formation of concepts: point, line, surface. Relations of incidence: point–line, point–surface, line–line.
- Straight lines and curved lines (as projective concepts). Open curves and closed curves (topological arc and topological circle). Recognition of the fundamental geometric shapes: segment, circle, rectangle, quader (parallelepiped), cylinder, ball. Role of these shapes in comparison activities: longer, wider, higher than, etc.

COMMENT. *All pregeometric concepts are at the sensory level inherent in the real world objects and iconic representations. Consequently, the learning process goes from observation to conception and a precise way of verbal expression should discriminate observable things from scientific concepts in their early state of creation. Representation of empirical situations by geometric drawings eliminates the existing noise and opens the road to abstraction. Thus, such activities help a child to organize his/her own thoughts about the structure and operations of the surrounding world.*

3. Analysis of primary school arithmetic

Nowadays arithmetic books are seen to be overcrowded by elaborate adorning. Charmed by their beauty, we could easily fail to notice how an organized exposition of arithmetic has almost disappeared from them. And nothing retrograde exists if we evoke the following didactical credos of the 19th century Pestalozzi's followers:

- The ultimate aim of arithmetic teaching is creation of abstract concepts.

- The concept of number must be formed on the ground providing meaning and evidence.
- That ground must not be turned into mere playing. How right they were, then as now!

Expressing our opinion that children learn arithmetic with lightness and ease only when it is properly structured and worked out with great care and nicety of detail, we turn our attention back to didactical analysis.

3.1. Related history—a list of topics.

- Grammatical forms in contemporary languages indicating small number systems in preliterate cultures. Numbers as man's primary concepts. Babylonian number system. Egyptian number system. The Greek (Alexandrian) way of writing numbers. HinduArabic positional system and its spreading in Europe. Parallel development of ideas of numbers as ratios of integers and of magnitudes. Discovery of incommensurable magnitudes. Vieta's *logistica speciosa*. Descartes' "coordination" of the line. Development of arithmetic notations. Decimal fractions. Creation of symbolic algebra.
- Educational heritage: Grecian *logistica numerosa*, medieval scholasticism, Komensky's visual method, Pestalozzi's didactics—numbers, shapes and words as a basis of elementary education, von Rochow's graduation into number blocks, Grube's monographic method, etc.

COMMENT. *Natural numbers and their ratios have always been related to discrete realities (including all kinds of scales upon which quantities of measurable things are transposed). Meanwhile lengths, areas and volumes of purely geometric objects are conveyors of the meaning of real numbers.*

3.2. Itemization of arithmetic contents.

- Sets at sensory level. Assimilation of words "set", "element". Cantor's cognitive principle of invariance of number—"killing" of two kinds of noise present in observable collections of things: nature of elements to be counted and any kind of their organization.
- Building of number blocks (up to 10, 20, 100, ...). The role of simple arithmetic expressions denoting sums and products as a means of block extension. Blocks as systems of mutually related concepts. Specific didactical tasks naturally arising from this block building: procedural establishment of the main rules of arithmetic at the moments when they are particularly operative and their rhetorical expression, use of the rules as foundation and an explanation, through action, of arithmetic procedures, building of arithmetic tables, etc.
- The role of place holders in realizing procedural tasks of arithmetic. Interdependence of arithmetic operations. Letters in the role of unknowns. Simple equations and their solving based on interdependence of operations. Further steps in developing the idea of variable. Symbolic expression of main rules of arithmetic, etc.

COMMENT. *The whole process of learning arithmetic has to be seen in a way which starts with observable things, develops into ideographic iconic signs and ends* *with the symbolic codes of arithmetic. Quoting Husserl, natural numbers and four operations are not "readymade products", but they have to be synthesized in pupil's mind through diligently planned and performed activities. Each individual number is a concept in itself and, using operations, they become interrelated so forming systems (structures) of initial blocks. A deviant contemporary tendency of ignoring the fundamental role of these systems diminishes the understanding and leaves some subtle pieces of arithmetic skill to look after themselves.*

Up to now, we have tried to give a sketch, in no way complete, of the didactical analysis of the two big themes of primary school mathematics. Following a normal course of ideas, such themes as further extensions of number systems and their structuring, development of the key idea of variable, etc. should be treated. But now we will leave such considerations aside and turn our discussion to a particularly controversial theme—logic in school.

4. Basics of mathematical logic

Development of logical thinking has always been stressed as a main objective of mathematical education. Taken broadly this thinking includes ability to abstract, precise use of the learnt concepts and the skill to form and logically evaluate compound sentences. In everyday language, two sentences are combined together to form a third one by means of connective words "and", "or", "if ... then ... ". The logical function of these words is learnt spontaneously, speaking the language. In educational practice of some countries this traditional view of spontaneous assimilation of the logical function of connectives is still held. But with the complexity of sentences, expressing various mathematical conditions, particularly in symbolic form, a need for precise use of these words increases. The fact that in a natural language these connectives also link pairs of words or phrases as well as elliptic forms of sentences, often having different subjects, makes this need for precision a more essential didactical task.

As an innovation during the period of "New Maths" curricula, the basics of mathematical logic entered school curricula in a form found in scientific expositions. When still apparent in the school programmes, we can see these basics begin with truth tables and with the letters denoting propositions. Problems of the type "determine the truth value of" are given (and the students solve them easily). This situation becomes problematic when letters denoting propositions are replaced by statements which mean something. Then, for example, both implications

$$1 > 2 \implies 3 < 4, \qquad 1 > 2 \implies 3 > 4$$

are taken to be true and the rules then begin to appear to be arbitrary. Without a clear "cause–effect" relation, such compound statements are normally taken by an unsophisticated student to be absurd rather than true. Recalling that all compound sentences found in the contents of mathematics have rhetoric variables for their subjects (case of geometry) or they are predicate formulas involving one or more variables (case of algebra), we turn our attention to the major disadvantage of such approach to logic. Those sentences, which are "vehicles of a certain sort of meaning" (Russell's phrase), determine propositional functions having propositions for their values at each point of their domains of definition. Thus, the transition from propositions to propositional functions is a more abrupt semantic jump than that in algebra from numerical to literal expressions. Without an appreciation of the role of logic in dealing with sentences having variables for their subjects, logic in school might appear to be a useless topic. A formal exposition which starts with the truth tables is certainly not a proper didactical transposition of this theme.

It is usual to start developing logical thinking in this narrower sense, at the start of the Piagetian period of formal operations and should be sustained throughout this whole period. Despite the fact that there exist a great number of research papers treating the questions of learning the basics of logic in school, there does not exist a more or less widely accepted conception and plan of execution for logical ingredients in school mathematics courses. Such ingredients are implicit and inevitably present in the long-term tasks of school mathematics such as solving of equations and their systems, solving of inequalities, etc. It is within these tasks that the basics of mathematical logic should start being diligently elaborated. Omitting details and stages in elaboration, we will schedule the main ideas of a plan of exposition, using the abstract language of mathematical logic (and not its substitutes which would be more appropriate terms in a didactical transposition of this theme).

- Syntax of mathematical logic should be seen as a generalization of the algebraic language.
- Meaning of connective words "and", "or" and "not" should be related to the set theoretical operations "intersection", "union" and "complement".
- Only the sentences having the same subject should be connected to form compound sentences. (In the simplest case two such sentences are: x belongs to A, x belongs to B.)
- In the early stage of elaboration, the connective words "if and only if" and "if ... then ... " are interpreted as relations in the sets of propositional functions, not as logical operations. Then, by finding of the truth sets, these connectives are related to equality and inclusion of sets, respectively.
- All exposition should be involved in the substantive concerns of mathematics, serving also to their clarification.
- This is again an instance in which the notion of set proves as a serviceable tool for concept-creation.

Contrary to the basics of set theory, which have found their right place and a proper didactical transposition in the current school programmes, the basics of mathematical logic still wait for an adequate elaboration, which would develop gradually through a number of successive didactical transformations.

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