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EXPLORING NUMBER-PYRAMIDS IN THE SECONDARY SCHOOLS

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Abstract. Some properties of the so called number-pyramids are given and their use in teaching practice for secondary schools are presented.

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Key words and phrases: Number-pyramids.

Number-pyramids are well-known and a typical exercise for addition and subtraction in German primary schools. As you can see in Fig. 1 they are built up in such a way, that the sum of the value of two neighboring bricks results in the value of the brick lying above them. But if you have a look at German teaching practice and textbooks in the secondary level, the search for them is almost unavailing. Therefore, some questions and problems (in chronological order to the German curriculum) will be presented and discussed in this article by which algebraic structures and mathematical coherency could be discovered and pointed out.

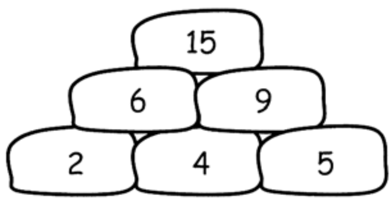


Fig. 1

Thoughts about elementary number theory concerning parity, divisibility or prime numbers could be motivated by the following questions:

Justify if number-pyramids could be completed

- *when only even (odd) numbers which must be divisible by 9 are allowed to be used;*
- *when only odd numbers are allowed to be used in the third row (counted from the bottom);*
- *when only prime numbers are allowed to be used.*

To give answers to these questions, pupils have to discover how and whether properties of numbers like parity and divisibility remain preserved if you add or subtract them.

E.g., due to the facts that the sum of two even numbers is always even and that 2 is the only even and at the same time the smallest prime number, 2 must be used as a brick in the base row of a number pyramid, if it is only allowed to complete it with prime numbers. Thus, number-pyramids with two bricks in the base row contain twin primes (Fig. 2). The question "*How many of those twin*

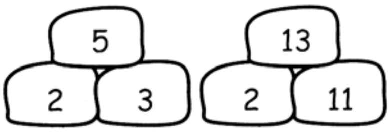


Fig. 2

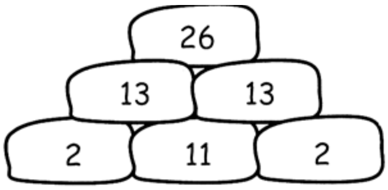


Fig. 3

primes exist?" gives a possibility to pick out an unsolved mathematical problem as a theme in mathematical education.

Number-pyramids with three or more bricks in the base row couldn't be completed solely with prime numbers, because the "third floor" contains at least a brick with an even number (e.g. Fig. 3).

It is worth to discuss fractional numbers in connection with number-pyramids. The completion of a number pyramid which only contains fractions in some bricks could be an exercise for the addition and subtraction of fractions and a discovery of rules at the same time. A possible task could be the question: "*Why are all numbers integers, if you multiply every number of the completed number pyramid (Fig. 4) by 30?*"

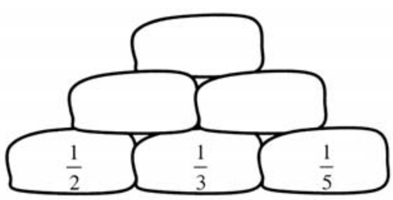


Fig. 4

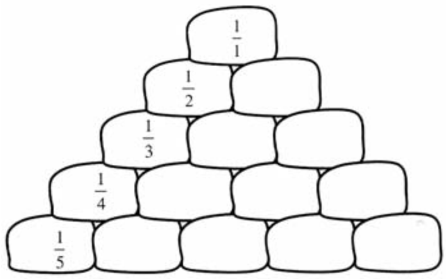
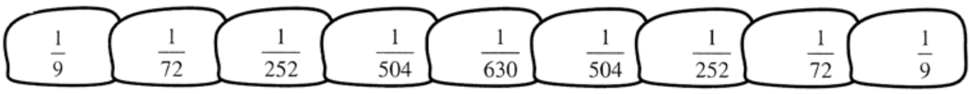


Fig. 5

Pupils of a mathematical working group (up to a 9th grade at a German gymnasium) got the task to complete the number pyramid in Fig. 5 and to search for rules. Almost all of the pupils who calculated accurately discovered the symmetry of the number pyramid. Furthermore, they went on calculating more rows than it was demanded. Afterwards they began searching for rules. Concerning the 9th row (counted from the top) a pupil calculated:



He noted the following multiplicative structure with a remark, that the numerator of the multiplying fraction increases in every step by one and that the denominator decreases in every step by one:

$$\frac{1}{9} \xrightarrow{\cdot\frac{1}{8}} \frac{1}{72} \xrightarrow{\cdot\frac{2}{7}} \frac{1}{252} \xrightarrow{\cdot\frac{3}{6}} \frac{1}{504} \xrightarrow{\cdot\frac{4}{5}} \frac{1}{630} \Rightarrow \text{Symmetry}$$

Afterwards he justified his discovery by comparing the results of the following multiplications with the numbers of the 9th row of his number pyramid:

$$\frac{1}{630} \cdot \frac{5}{4} = \frac{1}{504}, \quad \frac{1}{504} \cdot \frac{6}{3} = \frac{1}{252}, \quad \frac{1}{252} \cdot \frac{7}{2} = \frac{1}{72}, \quad \frac{1}{72} \cdot \frac{8}{1} = \frac{1}{8}.$$

For the diagonals of the number pyramid he discovered a resembling multiplicative structure, by which the numerator and the denominator of the multiplying fraction increase in every step by one.

A female pupil inverted the fractions of the completed number pyramid. Then she divided every number of a row by the first number of the row (Fig. 6) and noted, that the new number pyramid is the Pascal's triangle:

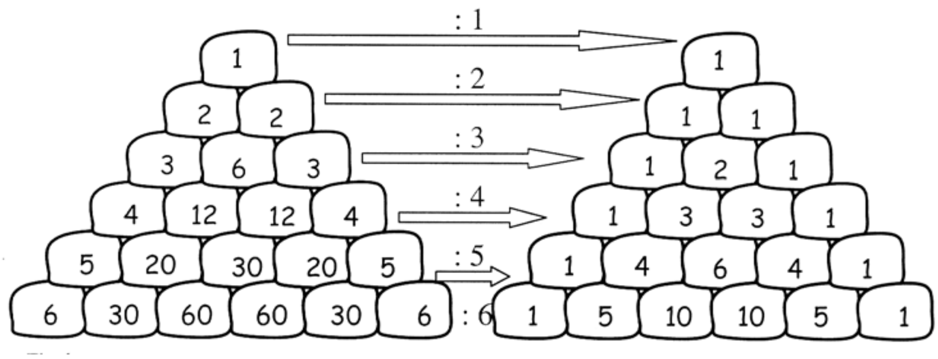


Fig. 6

Even the additive structures of percentages could be discovered with number-pyramids. The advisement whether you could add the percentage rates with equal percentage quotations could be motivated with the number pyramid in Fig. 7. In this context the converse problem whether you could add the percentage quotations with equal percentage rates (Fig. 8) could be discussed.

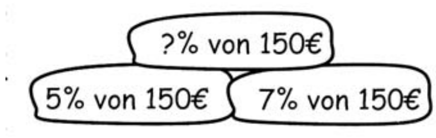


Fig. 7

Fig. 8

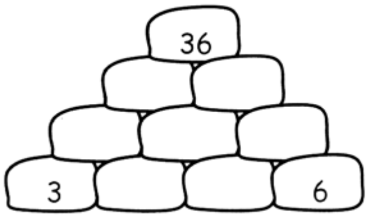


Fig. 9

Learning environments which use number-pyramids to introduce negative numbers and the concept of variables are described in detail (e.g.) in the following sources: [1], [2], [3] and [4]. An example for a possible learning environment could be presented as follows.

The pupils of a 7th grade at a German gymnasium got the following task:

1. *Try to complete the number-pyramid in many different ways (call every completed number-pyramid a solution).*
2. *What have the different solutions in common?*

While the pupils answered the tasks, different strategies were observable. Many pupils began finding solutions from the top of the number pyramid. They divided 36 into $18+18$ and tried to find suitable numbers for the rows below. None of them divided 36 non-symmetrically (e.g. $20+16$) or to non-integers (e.g. $18.5+17.5$).

Only a few pupils tried to start from the bottom of the number pyramid. They wrote two numbers between 3 and 6, completed the number-pyramid afterwards and compared the sum of the second row (counted from the top) with the top to see whether the chosen numbers in the base row are too small or too big.

A female pupil decided to write at first an integer number in the middle brick in the second row (counted from the bottom) to prove, if the number pyramid could be completed afterwards. She answered to the question why she had chosen this strategy with the statement, that always "nice numbers" are solutions of such tasks and due to this it is not important where to begin.

It was interesting, that none of the pupils used negative or non-integer numbers at all.

While searching for answers to the second question some pupils found the following similarities. The number in the middle of the second row (counted from the bottom) must be divisible by 3. Moreover, this number is always 9. Even the equivalent property, that the sum of the missing numbers in the base row is 9 was mentioned. It was unexpected, that pupils calculated and compared the sum of every row. They pointed out, that the first three sums 18, 27 and 36 are multiples of 9.

Afterwards the third task was presented to the class: "*How many different solutions there exist?*"

To answer this question the pupils regarded the property, that the sum of the missing numbers in the base row is always 9, as helpful. After a few minutes some pupils requested whether zero, negative or rational numbers are allowed to use, because 9 is also equal, e.g., to $-2+11$ or $9+0$. A few examples were completed on the blackboard. Some pupils noticed, that due to the conjecture of the constancy of the sum solutions could be found faster than via non-systematic trying. Due to this discussion the mentioned number of different solutions was quite different from each other, because some pupils accepted the use of negative numbers (and further on rational numbers) but for some of them only integers were real numbers[1]. Indeed some pupils miscounted their number of solutions but at last the pupils decided to accept three different cases, that (1.) all positive integers without zero could be used to complete the number-pyramid (8 solutions); that (2.) all positive integers included zero could be used (10 solutions); and that (3.) all rational numbers could be used (an infinite number of solutions).

Two male pupils found a trick to calculate the value of the brick in the middle of the second row (counted from the bottom). Their argumentation was to subtract the sum of the two corner bricks in the base row from 36 and then to divide the result 27 by 3, which is the wanted number 9. Their classmates asked them why it had to be divided by 3 (it was an interesting observation, that the classmates did not ask them why 9 had to be subtracted from 36). They argued, that they had considered the sum of the rows. One could imagine the sum of the missing bricks in the base row as an unknown number x. Then the sum of the base row equals $3 + x + 6$, thus $x + 9$. In the second row (counted from the bottom) the number x appears one time in the middle brick and once again one time due to the distribution of its two summands on the left and the right brick. Thus the sum of the second row is $2x + 9$. Therefore, (they now deduced without considering the number-pyramid) the sum of the third row is $3x + 9$, so you only have to calculate $36 - 9$ and to divide the result by 3.

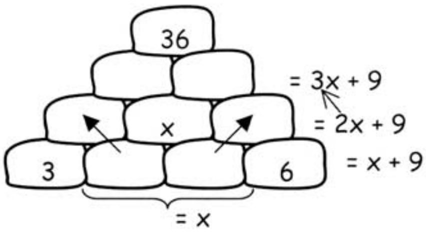


Fig. 10

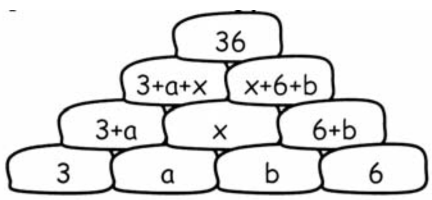


Fig. 11

This pragmatic part of their explanation was the subject of the following lesson. The pupils analyzed and developed the number-pyramid on the board (Fig. 11). Now they recognized, that x is only two times contained in the third row, but the sum of a and b, which one can find in this row, too, was the third missing x. The equivalence of $x = (36 - 9) : 3$ (based on the argumentation of the two pupils) and $3x + 9 = 36$ (as a solution of the analysis of the whole class) was now explained in two ways: On one hand pupils supposed to simplify $x = (36 - 9) : 3$ to $x = 9$. Then it only has to be proved that, if you insert 9 for x, then $3x + 9 = 36$.

On the other hand pupils argued, that this calculation is not necessary because everyone could see that $x = (36 - 9) : 3$ and $3x + 9 = 36$ means the same. Because if you want to know what x is in $3x + 9 = 36$ you only have to do what the two boys supposed to do: subtract 9 from 36 and divide the result by 3.

[1]This reminded me in a sense of the historical discussion whether negative numbers have to be accepted or not.

One must suggest, that most of the pupils in this class are very interested in mathematics. But nevertheless it is astonishing how many ideas you could find concerning the concept of variables and equalities in the mind of the pupils without picking it out as a central theme.

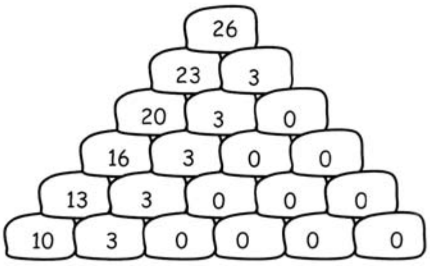


Fig. 12. Value of the left brick = 3· number of its row +7

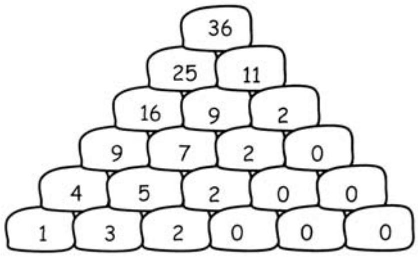


Fig. 13. Value of the left brick = number of its row ^{2}

A further field of problems are growth processes (concerning equidistant arguments). Linear growth is characterized by constant increment. A characteristic of quadratic growth is the constant increment of the increment. This principle could be continued further on for cubic, biquadratic and higher growth. In the upper secondary schools it will be developed in the context of the infinitesimal calculus and formulated by the proposition that the n-th derivation of a polynomial function with order n is a constant function. Considering linear, quadratic and cubic growth the bricks in the left border diagonal in Figs. 12, 13 and 14 describe the respective values.

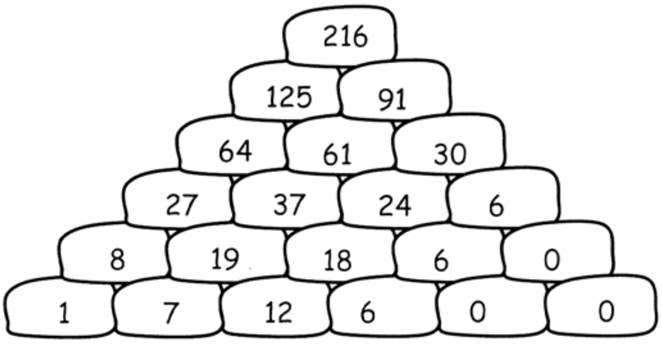


Fig. 14

Fig. 14. Value of the left brick = number of its row ^{3}

If there are not enough rows to discover the arithmetical structure, e.g. for growth of higher order, it is possible and easy to add some diagonals on the right side of the number-pyramid. Different kind of growths could be combined and the underlying algebraic structure could be analyzed in different grades at different levels in the same context.

Number-pyramids are also a source for tasks concerning equations and systems of equations. Teacher students got the number-pyramid in Fig. 15 with the task to find the number of solutions in **N** and **Z** justified by algebraic considerations. In the following some solutions are presented.

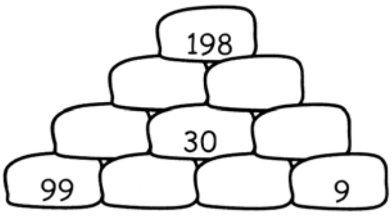


Fig. 15

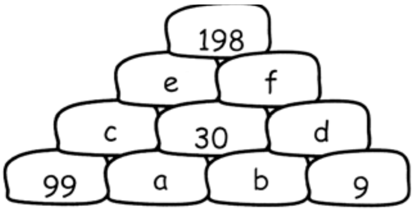


Fig. 16

A female student completed the number pyramid as in Fig. 16 and noted:

$$\begin{array}{ccc} a+b=30 & 30+c=e & e+f=198 \\ a+99=c & 30+d=f & \\ b+9=d & & \end{array}$$

She concluded

$$\begin{array}{rcccl} 30+c=e & \Longrightarrow & 30+a+99=e & \Longrightarrow & 129+a=e \\ 30+d=f & \Longrightarrow & 30+b+9=f & \Longrightarrow & 39+b=f \\ e+f=198 & \Longrightarrow & 129+a+39+b=198 & \Longrightarrow & a+b=30 \end{array}$$

Afterwards she noted: "The only assumption to complete the number-pyramid is that $a+b=30$! Thus there are several solutions in **N** and **Z**."

In fact her conclusion $a+b=30$ was right and furthermore she considered that the constraint $e+f=198$ must be fulfilled, too, but her suggestion concerning the number of solutions and the complexity of her solution is remarkable.

Another female student completed the number pyramid as it is presented in Fig. 17. Below she noted:

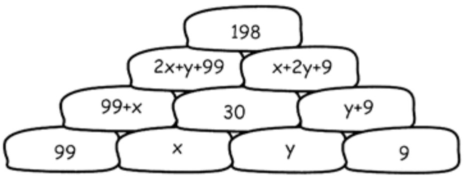


Fig. 17

"
$$\begin{aligned} 198 &= 2x+y+99+9+2y+x \\ 198 &= 3x+3y+108 \\ 90 &= 3x+3y \end{aligned}$$

$30 = x + y \leftarrow$ the only constraint which must be fulfilled! In $\mathbf{Z}$ there is an infinite number of solutions." Then she noted: "Solutions in $\mathbf{N}$:

$$0 + 30 = 30, \quad 1 + 29 = 30, \quad 2 + 28 = 30, \quad \ldots, \quad 15 + 15 = 30\text{''}$$

She concluded that there are 15(!) different solutions. A reflection about the commutativity of the summands was missing too. Even further solutions of teacher students contained errors concerning the number of different solutions or did not reflect if the constraints $30 = x + y$ and $198 = 2x + y + 99 + 9 + 2y + x$ (see Fig. 17) were consistent. In the discussion about the solutions with the teacher students they reflected that they learned the concept of equations and systems of equations isolated and insufficient embedded in context.

The strategy of factorisation is helpful even beyond the binomial formula and could be motivated with the task: "*Complete the following number-pyramids and search for relations between the corner bricks and the bricks between them.*"

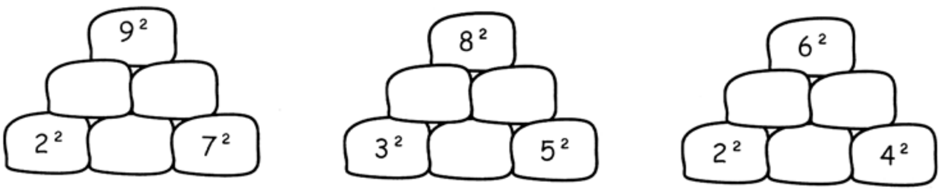


Fig. 18

This leads to the result:

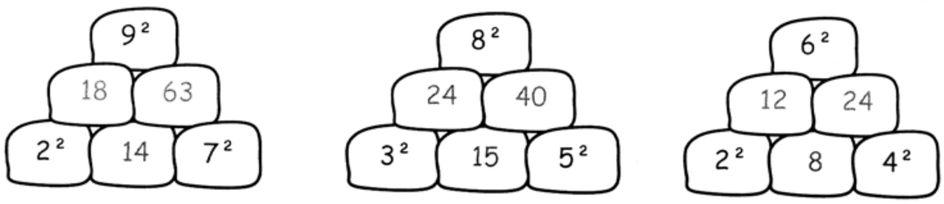


Fig. 19

Pupils could find:

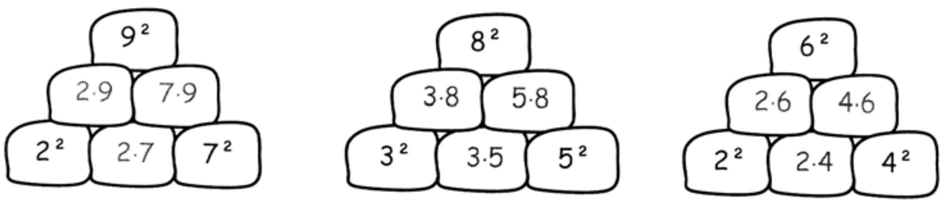


Fig. 20

With the distributive law and the binomial formula they could prove the generality of the number-pyramid in Fig. 21. This new knowledge could be used to make predictions concerning higher powers in analogy (Fig. 22).

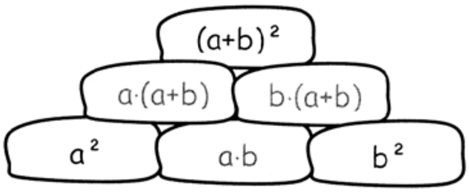


Fig. 21

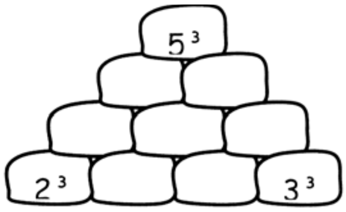


Fig. 22

In mathematical education normally the binomial formula is used to demonstrate that to square is not distributive concerning the addition. This property could be picked up in the context of number-pyramids by the following task: "*Find number-pyramids which only contain square numbers.*"

For number-pyramids with two bricks in the base row it is obvious that Pythagorean triples are needed (Fig. 23). For number-pyramids with three bricks in the base row 6 square numbers are required, where each triple must be a Pythagorean triple. It is not obvious that a 6-tuple of such square numbers exists at all.

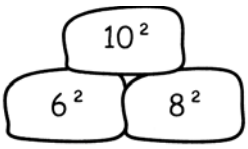


Fig. 23

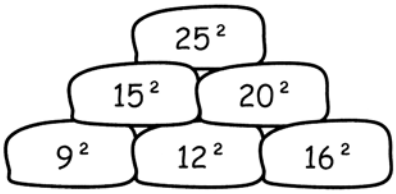


Fig. 24

Trying or programming could be a method to find such 6-tuple (e.g. Fig. 24). By comparing the bases of the numbers in Fig. 24 with the Pythagorean triple (3, 4, 5) and factoring (Fig. 25), a general structure could be concluded (Fig. 26), if (a, b, c) is a Pythagorean triple.

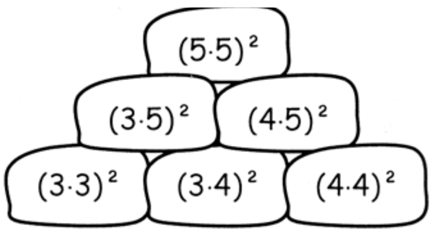


Fig. 25

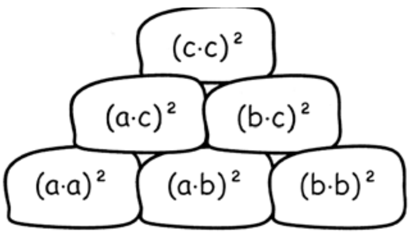


Fig. 26

With this method "square number-pyramids" could be created by multiplying ("stretching") a Pythagorean triple (a, b, c) by each of its terms a, b and c.

The geometrical translation of this task shows the affinity of the group of Pythagorean theorems, because right-angled triangles are sought, whose side lengths, altitude and the sections of the hypotenuse are integers (Fig. 27).

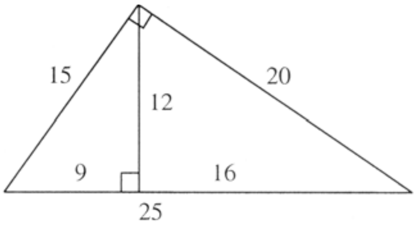


Fig. 27

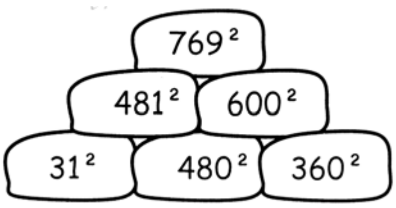


Fig. 28

But there are also "square number-pyramids" (Fig. 28) in which the Pythagorean triples are not stretched versions of the same basic triple, e.g. $(31, 480, 481)$, $(480, 360, 600)$ and $(481, 600, 769)$. It is unknown how many of those Pythagorean "triple-triple" exist and whether there is an underlying structure[2].

Even square roots are not distributive concerning the addition, either. In the majority of cases only the fact that the equality $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ is not valid in general will be a topic in mathematics education. "*Check, if the number-pyramid is completed correctly*" (Fig. 29). "*Correct if necessary and complete.*"

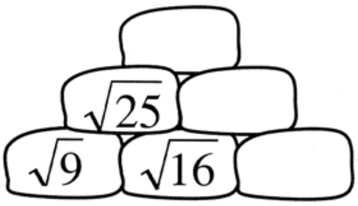


Fig. 29

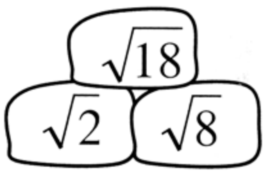


Fig. 30

These tasks allow pupils to give a lot of possible correct answers. In particular pupils are calculating while they are examining and vice versa.

It is also an interesting question, if number-pyramids could be completed only with square roots whose radicants are integers and non-square. Let $\sqrt{a}$ and $\sqrt{b}$ be the values of the bricks in the base row and $\sqrt{c}$ the value of the brick above them, then $\sqrt{a} + \sqrt{b} = \sqrt{c}$ must be valid. This implies $a + b + 2\sqrt{ab} = c$, so if $a \cdot b$ is a square number the number pyramid could be completed with non-square integers (consider Fig. 30 as an example). In teaching practice square roots usually are to be added by factoring the radicants (e.g. $\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4 \cdot 2} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2} = \sqrt{9 \cdot 2} = \sqrt{18}$).

If pupils discover the structure of such number-pyramids, they can check with this "square-number-criteria", whether square roots could be simplified when they should be added. At the same time they have got an alternative to add square roots (e.g. $\sqrt{2} + \sqrt{8} = \sqrt{2 + 8 + 2\sqrt{16}} = \sqrt{18}$).

[2]In particular these "triple-triple" cannot be interpreted geometrically in the mentioned way.

Logarithms, the great idea of John Napier to reduce the multiplication of two numbers to the addition, could be considered in the context of number-pyramids, too. The uncompleted number pyramid in Fig. 31 is a possibility to discover and to practice this property.

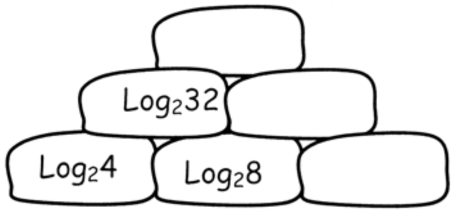


Fig. 31

In the upper secondary schools number-pyramids are an adequate substitute to clarify and to present an alternative example for a vector space. Therefore, it has to be clarified how number-pyramids (with the same number of bricks in the base row) have to be added and multiplied by a scalar (see Fig. 32).

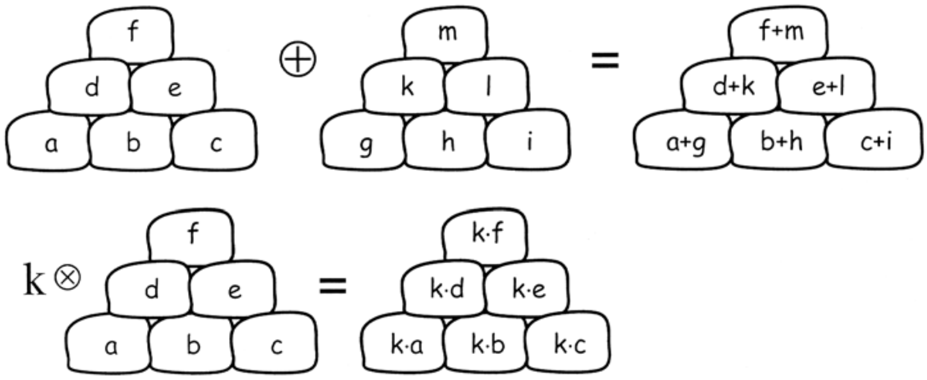


Fig. 32

Besides the verification of the closedness of the addition and the scalar multiplication the question concerning the dimension of the vector space of number-pyramids (e.g. with three bricks in the base row) should be discussed.

These mentioned examples are only a few possibilities to arrange learning environments with which (standard-) subjects of mathematics in the secondary schools could be imparted in the context of number-pyramids. For further ideas and suggestions (especially for those who clarify the conjecture, that there is an infinite number of solutions for the non-stretched square-number pyramid), please send an e-mail to: `jan.mueller@math.uni-dortmund.de`.

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