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EXTREMAL PROBLEMS—PAST AND PRESENT

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Abstract. The evolution of a phragment of the theory of extremal problems, the necessary conditions of extremum, is explained. Four problems of Fermat, Lagrange, Euler and Pontryagin are presented and four classical examples of Euclid, Kepler, Newton and Bernoulli are solved.

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1. Introduction

There are at least three reasons for solving extremal problems. The first one is pragmatic: it is typical for mankind to search for the best way of using its resources, and that is why a lot of problems of maxima and minima arise in the economics, in solving technical questions, in managing various processes. The second reason comes from the properties of the world around us: a lot of natural laws are explained by extremal principles. Finally, the third reason is man's curiosity, his wish to fully understand something.

Let us state four famous examples.

1. Euclid's problem. Euclid in his "Elements" (IVth century B.C.) states a solution of the following extremal problem:

Inscribe a parallelogram $ADEF$ of maximal area into the given triangle ABC (Fig. 1). This problem was not motivated by an application of any kind, and it does not explain any natural phenomenon. It was just an interesting geometrical problem.

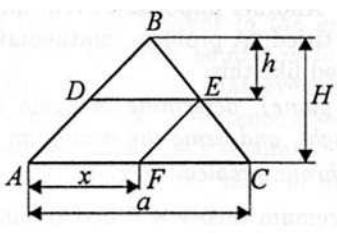

Fig. 1

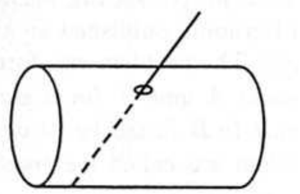

Fig. 2

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2. KEPLER'S PROBLEM. In 1613 Kepler was about to get married and so he was concerned with the organization of his household. That was a year when the crop of vineyards in his surrounding was very good and there were a lot of wine barrels that were carried near his home in Linz on Rhine. Kepler asked for some barrels to be brought to his yard, and so it was done. Then came a grader who determined the volume of barrels in a very simple manner. He put a measure-stick into the hole on top of the barrel as in Fig. 2, looked at the part of the stick which was red and immediately told the price. It seemed odd to Kepler: the barrels were of different kinds, and the way of measuring was always the same. The curiosity led Kepler to investigate the problem mathematically. As a result he made several crucial steps in the birth of integral calculus, he described various methods of calculation of areas and volumes and he also solved several extremal problems. He wrote about this in his book "Stereometry of Wine Barrels". He also solved there the following problem:

Inscribe a rectangular parallelepiped of maximal volume into the given sphere.

We can see that this problem was also a result of curiosity and the wish to fully understand a certain phenomenon.

3. NEWTON'S AERODYNAMICAL PROBLEM. The greatest scientific work of Newton, "The Mathematical Principles of Natural Philosophy" appeared in 1687. There was a problem of technical nature in this book. Newton posed and solved a problem about *a rotating body with the given width and height, and giving the smallest resistance in a viscous fluid* (Fig. 3). He added that his solution might be "used for constructing ships"—here is an example of a problem with pragmatic nature.

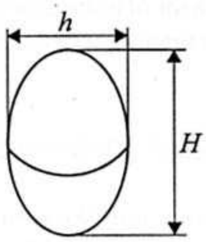

Fig. 3

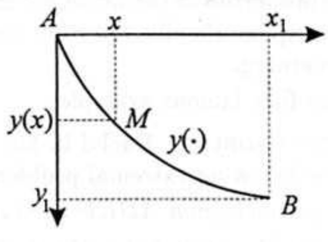

Fig. 4

4. THE BRACHISTOCHRONE PROBLEM. Another important event happened in 1696. Iohann Bernoulli published an article titled "A problem, mathematicians are called to solve". The problem was formulated like this:

Given points A and B (in a vertical plane), determine the path a body M descends from A to B forced by its own weight, and using the minimum of time.

This problem was called *the brachistochrone problem* (Fig. 4).

Here, four results from the general extremum theory will be explained, which are connected with four mathematicians: P. Fermat (1601–1665), J. Lagrange (1736–1813), L. Euler (1707–1783) and L. S. Pontryagin (1908–1988). All the concrete examples we have mentioned before will be solved.

2. Problems without constraints. A theorem of Fermat (1638)

Let f be a function of one variable (or, we can say, let f be defined on the real axis $\mathbf{R}$; then we write $f\colon \mathbf{R} \to \mathbf{R}$). Consider the problem: *Find the extremum (i.e., maximum or minimum) of function f without constraints.* Formally, we will write

$$(P_1) \qquad f(x) \to \text{extr}.$$

We will assume the function f to be differentiable. Recall the meaning of this. Suppose that the increment $f(\tilde{x}+x) - f(\tilde{x})$ can be written as the sum of the linear expression ax and a remainder $r(x)$, where $r(x)$ is "small compared with x". More precisely, let

$$f(\tilde{x}+x) = f(\tilde{x}) + ax + r(x), \quad \text{where } \lim_{x\to 0} \frac{|r(x)|}{|x|} = 0.$$

The linear function $y = ax$ is the *main linear part of the increment.* Number a is called *the derivative of function f at the point $\tilde{x}$*. It is denoted by $f'(\tilde{x})$. The following theorem is valid.

THEOREM 1. (Fermat) *If the function f is differentiable at the point $\tilde{x}$ which is a solution of problem (P_1), then*

$$(1) \qquad f'(\tilde{x}) = 0.$$

HISTORICAL COMMENT. Fermat did not know the concept of derivative, but actually (in his letter to Robervall and Mersenne in 1638, who were used by French scientists of the time for scientific correspondence—journals did not exist) he literally explained the idea of "the main linear part" of a function and said that it has to be equal to zero.

The concept of derivative was introduced by Newton and Leibniz. To Newton the derivative was the measure of velocity of variation of a process. The result of Theorem 1 was expressed by him as: "*at the moment when a quantity attains its maximum or minimum, it does not flow, either forwards, or backwards.*"

To Leibniz, the derivative was the slope of the tangent. So, in his words, Fermat's theorem says that "*the tangent to the graph of a function in an extremal point has to be horizontal*" (Fig. 5). Notice that even Kepler in his "Stereometry" had a sentence also expressing the essence of Fermat's theorem. He wrote that "*on both sides of the place of maximum, decreasing is not essential.*"

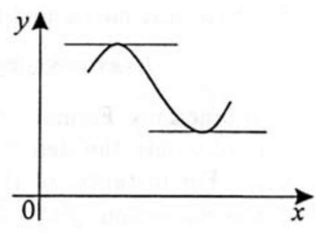

Fig. 5

Notice also that the equality $f'(\tilde{x}) = 0$ is a necessary, but not a sufficient condition for extremum. Point 0 is neither a point of maximum, nor a point of minimum, for the function $g(x) = x^3$, but $g'(0) = 0$.

Let us solve the Euclid's problem using Fermat's theorem. Let us look again to Fig. 1, where $a = AC$, $x = AF = DE$, H is the altitude of ABC and h the altitude of DBE. Using the similarity of triangles DBE and ABC, we obtain that $\dfrac{x}{a} = \dfrac{h}{H}$. The area of the parallelogram is

$$x(H-h) = \frac{H}{a}x(a-x).$$

So, the problem reduced to the problem of finding the maximum of the function $f_0(x) = x(a-x)$ (we neglected the constant factor H/a), with the constraint $0 < x < a$. But we can neglect this constraint, too, and we can consider a problem without constraints

$$f_0(x) = x(a-x) \to \max.$$

At the point of maximum, the equality $f'(\tilde{x}) = 0$, i.e., $\tilde{x} = a/2$ has to take place. And then

$$f_0(\tilde{x}+x) = \left(\frac{a}{2}+x\right)\left(\frac{a}{2}-x\right) = \frac{a^2}{4} - x^2 = f_0(\tilde{x}) - x^2,$$

i.e., f_0 attains its maximum at the point $\tilde{x}$, without any constraints, and, a fortiori, with our constraint. We have solved the problem—*point F has to be the midpoint of segment AC.*

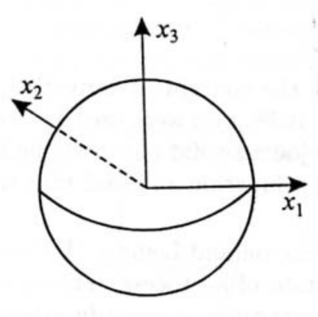

Fig. 6

Let us continue by considering extremal problems for functions of several variables. Consider the Kepler's problem. Suppose that we have constructed orthogonal axes, and denote variables by x_1, x_2 and x_3 (Fig. 6). Then the sphere with radius 1 can be written as

$$x_1^2 + x_2^2 + x_3^2 - 1 = 0.$$

Let a vertex of a parallelepiped lying on the sphere has coordinates (x_1, x_2, x_3) and denote it simply by x. Then the volume of the parallelepiped is equal to $8|x_1x_2x_3|$.

We have just mentioned two examples of functions of three variables:

$$f_1(x) = 8x_1x_2x_3 \quad \text{and} \quad f_2(x) = x_1^2 + x_2^2 + x_3^2 - 1.$$

For such functions, Fermat's theorem has the same formulation as for functions of one variable, only the derivative is now not a single number, but a collection of numbers. For instance, in the case of three variables, if $x = (x_1, x_2, x_3)$, denote by $|x|$ the expression $\sqrt{x_1^2 + x_2^2 + x_3^2}$. Let f be a function of three variables (then we write $f\colon \mathbf{R}^3 \to \mathbf{R}$), and let the increment $f(\tilde{x}+x) - f(\tilde{x})$ at point $\tilde{x}$ can be represented as the sum of the linear part $a_1x_1 + a_2x_2 + a_3x_3$ and the remainder $r(x)$, small when compared with x. More precisely, let

$$f(\tilde{x}+x) = f(\tilde{x}) + a \cdot x + r(x),$$

where $a = (a_1, a_2, a_3)$ is a collection of three numbers, $a \cdot x$ denote the "scalar product" of a and x, i.e., $a \cdot x = a_1x_1 + a_2x_2 + a_3x_3$, and $\lim\limits_{|x|\to 0} \dfrac{|r(x)|}{|x|} = 0$. Then we say that function f is *differentiable* at $\tilde{x}$ and that a is the derivative of f at the point $\tilde{x}$; denote it by $f'(\tilde{x})$. The derivative $f'(\tilde{x})$ is the collection of three numbers $(f'_{x_1}(\tilde{x}), f'_{x_2}(\tilde{x}), f'_{x_3}(\tilde{x}))$, where $f'_{x_1}(\tilde{x})$ is the derivative at zero of the function of one variable $g_1(x) = f(\tilde{x}_1 + x, \tilde{x}_2, \tilde{x}_3)$; similarly $f'_{x_2}(\tilde{x})$ and $f'_{x_3}(\tilde{x})$ are defined.

Consider the problem without constraints

$$(P_1') \qquad f(x) \to \text{extr},$$

where $x = (x_1, x_2, x_3)$ or even $x = (x_1, x_2, \dots, x_n)$ (function of n variables). The following theorem is valid

THEOREM $1'$. *If the function f is differentiable at the point $\tilde{x}$ and this point is a solution of problem (P_1'), then $f'(\tilde{x}) = 0$ (or, in the three-dimensional case, $f'_{x_1}(\tilde{x}) = f'_{x_2}(\tilde{x}) = f'_{x_3}(\tilde{x}) = 0$).*

But there are few interesting problems without constraints of this kind. As a rule, problems with constraints are more important (Kepler's problem is one of them). A general method for solving problems with constraints belongs to Lagrange.

3. Finite-dimensional problems with constraints. Lagrange's multipliers rule (1801)

Consider the problem

$$(P_2) \qquad f_0(x) \to \text{extr}, \quad f_1(x) = 0,$$

where f_0 and f_1 are functions of n variables: $x = (x_1, x_2, \dots, x_n)$ (then we write $f\colon \mathbf{R}^n \to \mathbf{R}$). There is a way of solving problems of the kind (P_2), belonging to Lagrange, by which one has to form the function $L(x, \lambda) = \lambda_0 f_0(x) + \lambda_1 f_1(x)$ with indefinite multipliers λ_0 and λ_1 (this function is called *Lagrange function* and $\lambda = (\lambda_0, \lambda_1)$ is a collection of *Lagrange multipliers*) and "*then search for maxima and minima*—as Lagrange wrote—*as if the variables were independent*", i.e., one has to apply Fermat's theorem to the problem $L(x, \lambda) \to$ extr without constraints. More precisely, the following theorem is valid.

THEOREM 2. (Lagrange's multipliers rule) *Let functions f_i be continuously differentiable in a neighbourhood of $\tilde{x}$ and this point is a solution of problem (P_2). Then there is a collection of Lagrange multipliers $\lambda = (\lambda_0, \lambda_1)$, distinct from zero $(|\lambda_0| + |\lambda_1| \neq 0)$, such that*

$$(2) \qquad L'_x(\tilde{x}, \lambda) = 0, \quad \textit{i.e.} \quad L'_{x_1}(\tilde{x}, \lambda) = 0,\ L'_{x_2}(\tilde{x}, \lambda) = 0,\ \dots,\ L'_{x_n}(\tilde{x}, \lambda) = 0.$$

Let us solve Kepler's problem by Lagrange's method. It can be formalized as

$$(i) \qquad f_0(x) = x_1x_2x_3, \quad f_1(x) = x_1^2 + x_2^2 + x_3^2 - 1 = 0$$

(we neglected the factor 8). Lagrange's function is

$$L(x, \lambda) = \lambda_0 x_1 x_2 x_3 + \lambda_1 (x_1^2 + x_2^2 + x_3^2 - 1).$$

Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ be a solution; then $\tilde{x}_i \neq 0$, $i = 1, 2, 3$ (there is no parallelepiped if some of them is zero). By Lagrange's theorem, $L'_{x_1}(\tilde{x}, \lambda) = 0$, and so $\lambda_0 \tilde{x}_2 \tilde{x}_3 + 2\lambda_1 \tilde{x}_1 = 0$, or, multiplying by $\tilde{x}_1$, $\lambda_0 \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 + 2\lambda_1 \tilde{x}_1^2 = 0$. Analogously, from $L'_{x_2}(\tilde{x}, \lambda) = 0$ it follows that $\lambda_0 \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 + 2\lambda_1 \tilde{x}_2^2 = 0$ and from $L'_{x_3}(\tilde{x}, \lambda) = 0$ it follows that $\lambda_0 \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 + 2\lambda_1 \tilde{x}_3^2 = 0$. If we let $\lambda_1 = 0$, we would get that $\lambda_0 = 0$, and this contradicts the condition of the Theorem that not both of Lagrange multipliers can be equal to zero. We see that

$$\tilde{x}_i = \frac{\lambda_0 \tilde{x}_1 \tilde{x}_2 \tilde{x}_3}{2\lambda_1}, \quad i = 1, 2, 3.$$

This means the only possible solution is the cube for which $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = \pm 1/\sqrt{3}$.

It is proved in Mathematical Analysis that a solution of problem (i) exists, but then it must satisfy Theorem 2 and there are eight solutions of equation (2) for this problem, and all of them are vertices of the cube. Consequently, *the cube is the solution of Kepler's problem.*

Historical comments. Lagrange described his solution in the book "Theory of Analytic Functions" in 1801.

4. Problems of Calculus of Variation. Euler's theorem (1744)

It can appear to be strange that, when stating the names of mathematicians that contributed to the extremum theory, I mentioned Lagrange before his elder colleague Euler, as if I violated chronological order of things: multipliers rule is dated 1801, and Euler's equation 1744. The reason is that the extremum theory really made an unexpected jump, and it passed from functions of one variable straight to functions whose arguments are curves, i.e., to functions with infinite number of variables. Let us make it more clear on the example of brachistochrone.

Let us direct the Oy-axis vertically down, put the point A into the origin, and let coordinates of point B be (x_1, y_1) (Fig. 4). Let $y(\cdot)$ be a certain curve ($y(\cdot)$ is the symbol for the function itself, $y(x)$ is the value of this function at point x). According to Galileo's law, a body with mass m, descending along the curve $y(\cdot)$, starting from the origin by the gravitational force, attains at the point $M(x, y)$ the velocity $\sqrt{2gy(x)}$, regardless of the mass m and the path it followed to come to point M. Consequently, when moving along the curve $y(\cdot)$, from the point $M(x, y(x))$ to the point $(x + dx, y(x + dx))$, for small dx, the path it passed is approximately equal to

$$\sqrt{dx^2 + dy^2} = \sqrt{dx^2 + y'^2(x)\, dx^2} = \sqrt{1 + y'^2(x)}\; dx,$$

and so, the time dt for passing this path is approximately equal to the ratio of the path and the velocity, i.e., $dt = \dfrac{\sqrt{1 + y'^2(x)}\; dx}{\sqrt{2gy(x)}}$. And so, the full time of passing the path from A to B is equal to

$$\int_0^{x_1} \frac{\sqrt{1+y'^2(x)}\,dx}{\sqrt{2gy(x)}}.$$

We have reformulated the brachistochrone problem: *one has to find the minimum of the stated integral (considering all curves $y(\cdot)$ satisfying the conditions $y(0)=0$, $y(x_1)=y_1$); in other words*:

$$\int_0^{x_1} \frac{\sqrt{1+y'^2(x)}\,dx}{\sqrt{y(x)}} \to \min, \qquad y(0)=0, \quad y(x_1)=y_1$$

(we neglected the factor $1/\sqrt{2g}$).

In the 1720's, there was a young man who started coming to lectures given by I. Bernoulli, and the lecturer immediately paid attention to him. The young man was Leonhard Euler. I. Bernoulli posed to Euler the problem of *finding the general method of solving problems analogous to the brachistochrone problem*, and Euler really did find such method. He generalized the brachistochrone problem in the following way. Let $f=f(x,y,z)$ be a function of three variables and $y(\cdot)$ a function, differentiable on the segment $[x_0,x_1]$. Then the number $\int_{x_0}^{x_1} f(x,y(x),y'(x))\,dx$ depends on the curve $y(\cdot)$. This is a "function of the function" (sometimes they called it, and some call it even now, a "*functional*"). Denote it by $J(y(\cdot))$. For example, in the brachistochrone problem, $f(x,y,z)=\dfrac{\sqrt{1+z^2}}{\sqrt{y}}$ (f does not depend on x).

Euler developed a method of solving problems like

$$(P_3) \qquad J(y(\cdot)) = \int_{x_0}^{x_1} f(x,y(x),y'(x))\,dx \to \text{extr}, \qquad y(x_0)=y_0, \quad y(x_1)=y_1.$$

Function f is called an *integrand* in problem (P_3). Problem (P_3) is called *the simplest problem of the Calculus of Variation.* We have

THEOREM 3. *If the function f is differentiable (as a function of three variables), and the function $\tilde{y}(\cdot)$ is a solution of problem (P_3), then the following differential equation is satisfied:*

$$(3) \qquad -\frac{d}{dx} f_z(x,\tilde{y}(x),\tilde{y}'(x)) + f_y(x,\tilde{y}(x),\tilde{y}'(x)) = 0.$$

Equation (3) is usually called the *Euler's equation.*

If f does not depend on x, then equation (3) has an integral

$$(3') \qquad \tilde{y}'(x) f_z(\tilde{y}(x),\tilde{y}'(x)) - f(\tilde{y}(x),\tilde{y}'(x)) = \text{const}.$$

Let us apply $(3')$ to the brachistochrone. We have

$$y' f_z - f = \frac{y'^2}{\sqrt{1+y'^2}\,\sqrt{y}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}} = -\frac{1}{\sqrt{y}\sqrt{1+y'^2}},$$

i.e., $\sqrt{y}\sqrt{1+y'^2} = \text{const} = \sqrt{C}$, and so $\sqrt{\dfrac{C}{y}-1} = y'$. Substitute $y = C\sin^2\dfrac{\tau}{2}$. Then, $\dfrac{dy}{dx} = C\cos\dfrac{\tau}{2}\sin\dfrac{\tau}{2}$. But,

$$\operatorname{ctg}\tau = \sqrt{\frac{C}{y}-1} = \frac{dy}{dx} = \frac{dy}{d\tau}\cdot\frac{d\tau}{dx},$$

and, consequently, $\dfrac{dx}{d\tau} = C\sin^2\dfrac{\tau}{2}$. So, we obtain a solution in the parametric form

$$x = \frac{C}{2}(\tau - \sin\tau), \quad y = \frac{C}{2}(1-\cos\tau).$$

This curve is called a cycloid. And so, *the brachistochrone appeared to be a cycloid.*

Historical comment. In 1744, Euler published his famous memoir "A method of finding curves having maximal and minimal properties, or a solution of the isoperimetric problem, taken in the widest possible sense", in which he developed the basics of the Calculus of Variation, and, particularly, introduced the Euler's equation.

5. Problems of optimal control. Pontryagin's maximum principle (1958)

More than two hundred years have passed since Euler's memoir was published, and the Calculus of Variation have almost reached its final form. Particularly, the following class of problems, similar to (P_2), has been investigated:

$$(P_3')\qquad \begin{aligned} I_0(y(\cdot)) &= \int_{x_0}^{x_1} f_0(x, y(x), y'(x))\,dx \to \min,\\ I_1(y(\cdot)) &= \int_{x_0}^{x_1} f_1(x, y(x), y'(x))\,dx = \gamma, \quad x(t_i) = y_i, \ \ i = 0, 1\end{aligned}$$

(problems like that where called "isoperimetrical in the widest sense" by Euler). The same idea of Lagrange that was considered earlier, can be applied to this problem. Namely, in order to solve (P_3'), one has to form the Lagrange function

$$L(y(\cdot),\lambda) = \int_{x_0}^{x_1} (\lambda_0 f_0(x, y(x), y'(x)) + \lambda_1 f_1(x, y(x), y'(x)))\,dx$$

and looking at the minimization problem of this function, one has to write down Euler's equation

$$(3')\qquad \begin{aligned} &-\lambda_0\frac{d}{dx}f_{0z}(x, y(x), y'(x)) + \lambda_0 f_{0y}(x, y(x), y'(x))\\ &\qquad -\lambda_1\frac{d}{dx}f_{1z}(x, y(x), y'(x)) + \lambda_1 f_{1y}(x, y(x), y'(x)) = 0,\end{aligned}$$

solve it, and try to satisfy the condition $I_1(y(\cdot)) = \gamma$. (Equations of the type $(3')$ are called *Euler-Lagrange equations*).

Let us show how these equations can be applied in the following example:

$$\int_0^\pi y^2\,dx \to \max, \quad \int_0^\pi y'^2\,dx = 1, \quad y(0) = y(\pi) = 0.$$

Here, equation ($3'$) has the form $-\lambda_1 y'' + \lambda_o y = 0$, or $y'' + \nu y = 0$, where $\nu = -\lambda_0/\lambda_1$. Boundary conditions are satisfied by the sequence $y_n(x) = \sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin nx}{n}$. The maximum value is attained for the function $y_1 = \sqrt{2/\pi}\,\sin x$.

And two hundred years later, during 1940's and 1950's, it was the need of Control Theory, Economics, Space Navigation, Military industry, that brought the necessity to make supplements to the theory of Calculus of Variation introducing new constraints—the constraints containing inequalities about variable control. When applied to problem (P_3), such constraints are imposed to the derivative of the function $y(x)$. We can write it down in the form $y'(x) \in U$, where U is a certain subset of $\mathbf{R}$ (say, finite segment $[a,b]$ or the semiaxis $\mathbf{R}_+ = \{x \geqslant 0 \mid x \in \mathbf{R}\}$, or even a finite set of points).

The first problem of optimal control was, without doubt, Newton's aerodynamical problem. In his "Mathematical Principles", he just stated the answer, without formalization and solution. Two of his contemporaries—I. Bernoulli and his student l'Hospital—formalized the problem and tried to solve it analytically. They directed the body along x-axis, but if they had directed it along y-axis, they would have come to an easier expression for the integrand, and they would have come to the problem

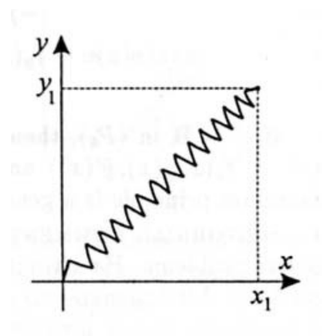

Fig. 7

$$(ii) \qquad I(y(\cdot)) = \int_0^{x_1} \frac{x\,dx}{1 + y'^2(x)} \to \min, \quad y(0) = 0, \quad y(x_1) = y_1.$$

"But this is just a problem of the Calculus of Variation—you might say—problem (P_3) with the integrand $f(x,z) = \dfrac{x}{1+z^2}$!" Moreover, there is something strange here: if one takes a cogged prophile with large slopes as in Fig. 7, then y'^2 can be very large and the integral in (ii) can become arbitrarily small.

One of serious specialists in Control Theory, L. Young, in his interesting book "Lectures on Calculus of Variation and Optimal Control Theory", stated a brutal objection to "illiterate" Newton. He wrote: "Newton formulated a variational problem about rotating body, causing the least resistance while moving in the gas. The physical law of resistance he applied was absurd, and as a result the given problem had no answer (the more cogged is the prophile, the resistance is smaller)". Alas, Young himself expressed here a stunning "illiteracy": he wrote a book on the Calculus of Variation and Optimal Control, but he did not realise that Newton's problem was not a variational problem, but a problem of optimal control, since *monotonicity* of the prophile, i.e., the inequality $y' \geqslant 0$, was to be understood. And Newton's solution was absolutely right!

The following problem is a particular, but important example of an optimal control problem:
(P_4)

$$I(y(\cdot)) = \int_{x_0}^{x_1} f(x, y(x), y'(x))\,dx \to \min, \quad y(x_0) = y_0, \quad y(x_1) = y_1, \quad y'(x) \in U,$$

where U is a certain subset of $\mathbf{R}$. In Newton's problem, $f(x,z) = \dfrac{x}{1+z^2}$, and $U = \mathbf{R}_+$. The following theorem is valid.

THEOREM 4. (Pontryagin's maximum principle for problem (P_4)) *If the function f is differentiable as a function of three variables, and $\tilde{y}$ is a solution of problem (P_4), then the following conditions are satisfied: there exists a function $p(\cdot)$ such that*

$$-p'(x) + f_y(x, \tilde{y}(x), \tilde{y}'(x)) = 0, \tag{4}$$

$$\max_{u \in U}(p(x)u - f_y(x, \tilde{y}(x), u)) = p(x)\tilde{y}'(x) - f_y(x, \tilde{y}(x), \tilde{y}'(x)). \tag{4'}$$

If $U = \mathbf{R}$ in (P_4), then relation $(4')$ can be differentiated by u. As a result, $p(x) = f_z(x, \tilde{y}(x), \tilde{y}'(x))$ and (4)-(4') is just Euler's equation. So, Pontryagin's maximum principle is a generalization of Euler's equation.

HISTORICAL COMMENTS. In the 1950's, L. S. Pontryagin got interested in control problems. He attracted his disciples—V. G. Boltyanskii, R. V. Gamkrelidze and E. G. Mishchenko—to problems of this kind. They formulated a special class of problems, called *problems of optimal control.* Necessary conditions for such problems were called the *Pontryagin's maximum principle.* It was V. G. Boltyanskii who proved Pontryagin's maximum principle for a wide class of optimal control problems.

6. Concluding remarks

Let us solve Newton's problem. The integrand f of this problem does not depend on y, and so (from (4)), $p(x) = \text{const} = p$ and the following maximum problem for a function of one variable has to be solved:

$$g(u,x) = pu - \frac{x}{1+u^2} \to \max, \quad u \geqslant 0 \tag{iii}$$

(here, u is the variable, and x is fixed). Clearly, $p < 0$, otherwise $\max g = \infty$.

If x is small, then maximum in (*iii*) is attained at zero. This is the case until the moment when the value at zero becomes equal to the second, positive maximum of this function. Critical point of function $\tilde{y}(\cdot)$, being the solution of Newton's problem, is characterized by the equation

$$p = \frac{2\tilde{y}(\xi)\xi}{(1+\tilde{y}'(\xi))^2}, \qquad \xi = \frac{\xi}{1+\tilde{y}'^2(\xi)} - py'(\xi).$$

It can be deduced that $\tilde{y}'(\xi) = 1$, $\xi = 2p$. And then, solving the equation $g'_x = 0$, one can obtain the solution in the parametric form

$$(iv) \qquad y = c\left(\ln\frac{1}{u} + u^2 + \frac{3}{4}u^4\right), \qquad x = c\left(\frac{1}{u} + 2u + u^2\right).$$

Constant c can be determined from the condition $y(x_1) = y_1$.

The curve (iv) is called *Newton's curve.* We see that the solution of Newton's problem has a corner point (Fig. 8). This has also provoked a doubt among many engineers: whoever has seen that a "ship" has a flat, and not sharpened edge in the front? And Newton, after calculating the corner-point angle to be 45°, said that even this note may be "not useless" for constructing ships! And once again, he appeared to be right. This "note" really appeared to be "not useless" for constructing "ships", namely supersonic planes and space crafts, moving in the space where the atmosphere follows the model of "Newton's thin surrounding". In fact, the optimal control was stimulated, among other things, by space problems. And when it came to construction of crafts having to fly with huge velocities and on great heights, Newton once again became one of the most cited authors.

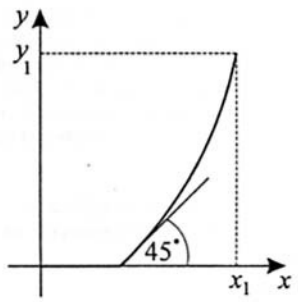

Fig. 8

In this article, we have told about the evolution of one phragment of the theory of extrema—about necessary conditions for the extremum. It is of interest to note that, in all the cases, two ideas where of the greatest importance—Kepler's idea (that in a neighbourhood of a maximum of a differentiable function, "decreasing is not essential", which brings us to Fermat's theorem), and Lagrange's idea that when dealing with problems with constraints, one has to consider the extremal problem for Lagrange's function "as if variables were independent". We have seen how these ideas work in finite-dimensional problems and in problems of the Calculus of Variation. With small changes, the same ideas can be applied to problems of optimal control: Pontryagin's maximum principle also follows Lagrange's idea.

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