THE TEACHING OF MATHEMATICS

2000, Vol. III, 2, pp. 115–119

DIVIDED DIFFERENCES

Vladimir Janković

Abstract. This article deals with Newton divided difference interpolation polynomials. Textbooks in numerical mathematics where such polynomials are studied usually put the emphasis on numerical problems which are solved using these polynomials. Here we show that such polynomials can also be useful in solving some algebraic problems. In order to present this concept we shall first define the divided differences in the new sense: the divided difference of order n is considered as a function of one variable and n parameters. After the concise presentation of the theory of divided differences, we shall solve the problem of interpolation by integer polynomials. At the end we give the solution of an interesting problem using this theory.

AMS Subject Classification: 00 A 35

Key words and phrases: Divided differences, interpolation polynomial.

Definition of divided differences and their properties

Let f be a complex function defined on a subset D of complex numbers. The *divided difference* of a function f at the point $x_0 \in D$ is the function $f(x_0, x)$ given by

$$f(x_0, x) = \frac{f(x) - f(x_0)}{x - x_0}.$$

Obviously the domain of this function is the set $D \setminus \{x_0\}$. If x_1 is a point from D different from x_0, the *second order divided difference* of the function f at points x_0 and x_1 is defined by

$$f(x_0, x_1, x) = \frac{f(x_0, x) - f(x_0, x_1)}{x - x_1}.$$

So, the second order divided difference of the function f at points x_0 and x_1 is the divided difference at the point x_1 of the divided difference at the point x_0 of f. This definition can be inductively extended to divided differences of arbitrary order in the following way: *the divided difference of order* $n+1$ of the function f at points $x_0, x_1, \dots, x_n$ is the divided difference at the point x_n of the divided difference at points $x_0, x_1, \dots, x_{n-1}$ of the function f, i.e.

$$f(x_0, x_1, \dots, x_n, x) = \frac{f(x_0, x_1, \dots, x_{n-1}, x) - f(x_0, x_1, \dots, x_{n-1}, x_n)}{x - x_n}.$$

The domain of the divided difference of f at points $x_0, x_1, \dots, x_{n-1}$ is the set $D \setminus \{x_0, x_1, \dots, x_{n-1}\}$.

THEOREM 1. *The following equality holds:*

$$f(x_0, x_1, \ldots, x_n) = \sum_{j=0}^{n} \frac{f(x_j)}{\prod\limits_{\substack{i=0 \\ i \neq j}}^{n} (x_j - x_i)}.$$

Proof. For $n = 1$ the assertion of the theorem holds because

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}.$$

Suppose that $n > 1$ and that the assertion is true for a positive integer $n-1$. Then we have

$$\begin{aligned}
&(x_n - x_{n-1}) f(x_0, x_1, \ldots, x_n) \\
&\quad = f(x_0, x_1, \ldots, x_{n-2}, x_n) - f(x_0, x_1, \ldots, x_{n-2}, x_{n-1}) \\
&\quad = \sum_{j=0}^{n-2} \frac{f(x_j)}{(x_j - x_n) \prod\limits_{\substack{i=0 \\ i \neq j}}^{n-2} (x_j - x_i)} - \sum_{j=0}^{n-2} \frac{f(x_j)}{(x_j - x_{n-1}) \prod\limits_{\substack{i=0 \\ i \neq j}}^{n-2} (x_j - x_i)} + \\
&\qquad + \frac{f(x_n)}{\prod\limits_{i=0}^{n-2} (x_n - x_i)} - \frac{f(x_{n-1})}{\prod\limits_{i=0}^{n-2} (x_{n-1} - x_i)} \\
&\quad = \sum_{j=0}^{n-2} \frac{(x_n - x_{n-1}) f(x_j)}{(x_j - x_n)(x_j - x_{n-1}) \prod\limits_{\substack{i=0 \\ i \neq j}}^{n-2} (x_j - x_i)} + \\
&\qquad + \frac{(x_n - x_{n-1}) f(x_{n-1})}{\prod\limits_{\substack{i=0 \\ i \neq n-1}}^{n} (x_{n-1} - x_i)} + \frac{(x_n - x_{n-1}) f(x_n)}{\prod\limits_{i=0}^{n-1} (x_n - x_i)} \\
&\quad = (x_n - x_{n-1}) \sum_{j=0}^{n} \frac{f(x_j)}{\prod\limits_{\substack{i=0 \\ i \neq j}}^{n} (x_j - x_i)},
\end{aligned}$$

so that dividing by $x_n - x_{n-1}$ we obtain that the assertion holds for the positive integer n. ∎

COROLLARY 1. *The function $f(x_0, x_1, \ldots, x_n)$ is symmetric.*

Divided differences of a polynomial

THEOREM 2. *If $f(x)$ is a polynomial of degree n, then $f(x_0, x)$ is a polynomial of degree $n-1$.*

Proof. $f(x) - f(x_0)$ is a polynomial of degree n which is divisible by the polynomial $x - x_0$, because it has a root x_0. Therefore $f(x_0, x) = (f(x) - f(x_0))/(x - x_0)$ is a polynomial of degree $n - 1$. ■

COROLLARY 2. *Divided difference of order $n+1$ of a polynomial of degree n is the polynomial identically equal to zero.*

Newton divided difference interpolation polynomial

THEOREM 3. *The following equality holds:*

$$\begin{aligned} f(x) = f(x_0) &+ f(x_0, x_1)(x - x_0) + \cdots + \\ &+ f(x_0, x_1, \ldots, x_n)(x - x_0)(x - x_1)\cdots(x - x_{n-1}) + \\ &+ f(x_0, x_1, \ldots, x_n, x)(x - x_0)(x - x_1)\cdots(x - x_{n-1})(x - x_n). \end{aligned}$$

Proof. According to the definition of divided differences we have

$$\begin{aligned} f(x) &= f(x_0) + f(x_0, x)(x - x_0), \\ f(x_0, x) &= f(x_0, x_1) + f(x_0, x_1, x)(x - x_1), \\ f(x_0, x_1, x) &= f(x_0, x_1, x_2) + f(x_0, x_1, x_2, x)(x - x_2), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f(x_0, x_1, \ldots, x_{n-1}, x) &= f(x_0, x_1, \ldots, x_{n-1}, x_n) + f(x_0, x_1, \ldots, x_n, x)(x - x_n). \end{aligned}$$

If we add the first equality multiplied by 1, the second equality multiplied by $x - x_0$, the third equality multiplied by $(x - x_0)(x - x_1)$, etc., and the last equality multiplied by $(x - x_0)(x - x_1)\cdots(x - x_{n-1})$ and if we cancel equal summands which appear on both sides, we obtain the desired equality. ■

COROLLARY 3. *The value of the polynomial*

$$f(x_0) + f(x_0, x_1)(x - x_0) + \cdots + f(x_0, x_1, \ldots, x_n)(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

at the point x_j is $f(x_j)$, $j = 0, 1, \ldots, n$.

COROLLARY 4. *If f is a polynomial of the degree n, then*

$$f(x) = f(x_0) + f(x_0, x_1)(x - x_0) + \cdots + f(x_0, x_1, \ldots, x_n)(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

The formula from Theorem 3 is called Newton divided difference interpolation formula.

Interpolation by a polynomial with integer coefficients

We consider the following problem:

Integers $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ are given, such that $x_i \neq x_j$ for $i \neq j$. Does there exist a polynomial p with integer coefficients satisfying the condition $p(x_j) = y_j$, $j = 0, 1, \ldots, n$?

Suppose that such polynomial exists. If we divide it by the polynomial $(x-x_0)(x-x_1)\cdots(x-x_n)$:

$$p(x)=(x-x_0)(x-x_1)\cdots(x-x_n)q(x)+r(x),$$

then the quotient $q(x)$ and the remainder $r(x)$ will also be polynomials with integer coefficients, because the leading coefficient of the polynomial $(x-x_0)(x-x_1)\cdots(x-x_n)$ is 1. The polynomial $r(x)$ satisfies the condition $r(x_j)=y_j$, $j=0,1,\dots,n$, and its degree is less than n, or it is equal to zero. In such a way this problem reduces to the following one:

> Integers $x_0,x_1,\dots,x_n$ and $y_0,y_1,\dots,y_n$ are given, such that $x_i\neq x_j$ for $i\neq j$. Does the interpolation polynomial $p(x)$ satisfying $p(x_j)=y_j$, $j=0,1,\dots,n$, have integer coefficients?

It turns out that Newton divided difference interpolation polynomial is the most suitable one for solving this problem. The following two theorems give the complete answer to the above question.

THEOREM 4. *Let $p(x)$ be a polynomial with integer coefficients and let x_0, x_1, $\dots$, x_n be different integers. Then $p(x_0,x_1,\dots,x_n)$ is integer.*

Proof. By induction on n it can be proved that $p(x_0,x_1,\dots,x_{n-1},x)$ is a polynomial with integer coefficients. The assertion of the theorem follows immediately from this fact. ∎

Note that for $n=1$ the assertion of the theorem reduces to: $[p(x_1)-p(x_0)]/(x_1-x_0)$ is integer, i.e. $x_1-x_0 \mid p(x_1)-p(x_0)$. This is a well known fact which can be used to solve many problems concerned with polynomials having integer coefficients.

THEOREM 5. *Let $p(x)$ be a polynomial of degree n and let $x_0,x_1,\dots,x_n$ be different integers. Coefficients of the polynomial $p(x)$ are integers if and only if the divided differences $p(x_0),p(x_0,x_1),\dots,p(x_0,x_1,\dots,x_n)$ are integers.*

Proof. In one direction the assertion follows from the previous theorem, and and the converse part follows from Newton divided difference interpolation formula:

$$\begin{aligned}p(x)=p(x_0)+p(x_0,x_1)(x-x_0)+\cdots+&\\ +\,p(x_0,x_1,\dots,x_n)&(x-x_0)(x-x_1)\cdots(x-x_{n-1}).\quad\blacksquare\end{aligned}$$

An interesting problem

At the end we show the possibility of application of the small theory developed in the preceding section by solving an interesting problem. In fact, the previous section was studied in order to obtain tools to solve the following problem, arising in the conversation of two friends:

Peter: How did you get such an ugly polynomial $f(x)$ with so many unknown coefficients? It looks terrible!

Paul: This is a polynomial with integer coefficients and one of its roots is equal to the number of years of my daughter Maria.

Peter: Anyway, it looks nasty. Let me find $f(7)$. Unfortunately, I obtained 77, not zero!

Paul: You don't know the age of Maria, but she is older than 7.

Peter: Let me try then with the age of my son John. Look, I obtained 85!

Paul: Watch out Peter, Maria is older than your son.

How old is John, how old is Maria?

Solution. Denote by j and m the ages of John and Maria, respectively. Conditions of the problem can be written in the following way:

$$f(m) = 0, \tag{1}$$

$$f(7) = 77, \tag{2}$$

$$f(j) = 85, \tag{3}$$

$$7 < j < m.$$

According to Theorem 5 there exists a polynomial f with integer coefficients satisfying conditions (1), (2) and (3) if and only if

$$\frac{77}{m-7}, \qquad \frac{77}{(m-7)(j-7)} - \frac{85}{(m-j)(j-7)} \tag{4}$$

are integers. From $m-7 \geq 2$ and $m-7 \mid 77$ it follows that m is one of the following numbers: 14, 18 and 84. Since

$$\frac{85}{m-j} = \frac{77}{m-7} - (j-7)\left(\frac{77}{(m-7)(j-7)} - \frac{85}{(m-j)(j-7)}\right),$$

it follows that $m-j \mid 85$. There are three cases to consider.

1. $m = 14$. From $0 < 14-j < 7$ and $14-j \mid 85$ it follows that $j = 9$. It can be checked immediately that for $m = 14$ and $j = 9$ the expression (4) is integer.

2. $m = 18$. From $0 < 18-j < 11$ and $18-j \mid 85$ we obtain that either $j = 13$ or $j = 17$. But it is seen immediately that for $m = 18$ and $j = 13$ or $j = 17$ the expression (4) is not integer.

3. $m = 84$. From $0 < 84-j < 77$ and $84-j \mid 85$ we obtain that $j = 67$, $j = 80$ or $j = 84$. Again we check immediately that if $m = 84$ and $j = 67$, $j = 80$ or $j = 84$, the the expression (4) is not integer.

Therefore the only solution of the problem is $m = 14$ and $j = 9$, i.e. Maria is 14 and John is 9 years old. ■

Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Beograd, Yugoslavia

E-mail: vjankovic@matf.bg.ac.yu