THE TEACHING OF MATHEMATICS
2000, Vol. III, 2, pp. 83–94

HOW TO MOTIVATE 10–18 YEARS OLD PUPILS TO WORK INDEPENDENTLY ON SOLVING MATHEMATICAL PROBLEMS

Judita Cofman

Abstract. The purpose of this article is to suggest a few problems for exploration which could motivate pupils to think and work independently on their solution. The article consists of three parts, describing experinces with pupils of different age groups: Part I describes a problem, related to solving the equation $ax+by=n$, with $a,b,n \in \mathbf{N}$ in non-negative integers x and y. This problem has been studied with pupils aged 14–15. Part II is devoted to sequences and combinatorial investigations of problems designed for 16–18 years old youngsters. In Part III the importance of introducing youngsters at an early age to independent work is underlined. This will be done by providing problems for children aged 10–13.

AMS Subject Classification: 00 A 35

Introduction

The following problem is included in the book “How to solve it: Modern Heuristics” of the American authors Z. Michalevicz and D. B. Fogel [1]:

> Two men, X and Y, meet on the street. X syas: “All three of my sons have their birthday this very day”. Y asks: “How old are they?” X replies: “The product of the ages of my sons is 36.” “Give me some more information” says Y. X points at the house next to them: “The sum of their ages is equal to the number of windows you see in this building”. Y thinks for some time and asks for an additional hint. “My oldest son has blue eyes” says X. “This is sufficient” exclaims Y and gives the correct ages of the sons.

A simple, common sense approach to solving this puzzle is the following: Since the ages of the sons are natural numbers with product 36, first of all one has to look at all possible ways of representing 36 as a product of three positive integers. There are eight such possibilities: 36,1,1; 18,2,1; 12,3,1; 9,4,1; 9,2,2; 6,6,1; 6,3,2 and 4,3,3. The sums of the factors are 38, 21, 16, 14, 13, 13, 11 and 10, respectively. There are only two sums: $9+2+2$ and $6+6+1$ equal to one another. Then the house next to Y must have 13 windows. The ages of the sons cannot be 6, 6 and 1 because in that case X would not have an oldest son. Hence the ages of the sons are 9, 2 and 2.

The authors of the book point out that although the puzzle is quite easy, many students of mathematics have difficulties with it. One of the reasons for such

This is one of the last articles by late professor Judita Cofman (1936–2001), a member of the Editorial Advisory Board of our journal. It is reprinted from the Proceedings of the 10th Congress of Yugoslav Mathematicians, Faculty of Mathematics, University of Belgrade, Belgrade 2001, pp. 21–32.

a failure is, according to Michalewicz and Fogel the following: Already at school, and also later at university, pupils and students are trained to solve problems by applying algorithms or formulae which they have learnt recently. When confronted with a problem, they automathically search for a suitable “ready made” algorithm, instead of finding their own way to a solution.

This situation has to be changed. Michalewicz and Fogel recommend that pupils should be regularly given problems whose solution requires independent, original thinking.

The purpose of this article is to suggest a few problems for exploration which could motivate pupils to think and work independently on their solution. The article consists of three parts, describing experinces with pupils of different age groups: Part I describes a problem, related to solving the equation $ax+by=n$, with $a, b, n \in \mathbf{N}$ in non-negative integers x and y. This problem has been studied with pupils aged 14–15. Part II is devoted to sequences and combinatorial investigations of problems designed for 16–18 years old youngsters. In Part III the importance of introducing youngsters at an early age to independent work is underlined. This will be done by providing problems for children aged 10–13.

Part I

The following set of problems was posed in a coursework for pupils aged 14–15:

Problem 1. Find the greatest natural number which *cannot* be written as a sum of numbers taken from the set $S = \{3, 5\}$.

Problem 2. Answer the same question as in Problem 1 if: a) $S = \{2, 7\}$; b) $S = \{6, 5\}$; c) $S = \{15, 14\}$.

Problem 3. Answer the same question as in Problem 1 if: a) $S = \{1, 9\}$; b) $S = \{4, 20\}$.

Problem 4. Compare the solutions of Problems 1–3 and describe your observations.

Problem 5. How should one choose natural numbers p and q such that the following property holds: There is no natural number not expressible as a sum with summands taken from the set $\{p, q\}$? Justify your answer.

Problem 6. Let p and q be coprime natural numbers greater than 1. Find the formula for the greatest natural number $g(p, q)$ which cannot be written as a sum with summands from $S = \{p, q\}$.

The coursework has been carried out by different groups of pupils during the last years. Its outcome is sketched in [2]: *Problem 1* was solved by most pupils as follows: By expection it was found out that the numbers 1, 2, 4 and 7 cannot be written as sums with summands 3 and 5, while 8, 9 and 10 are three consecutive numbers which are expressible as sums of 3 and 5. This implies that all natural numbers greater than 7, being of the form $8+3k$, $9+3k$ and $10+3k$ for $k = 0, 1, 2, \ldots$, can be written as sums of 3 and 5. Thus the greatest natural number not expressible as sum of elements from $S = \{3, 5\}$ is 7. Answers to parts a) and b) of *Problem 2* were also easily obtained by the same strategy; they were the numbers

5 and 19. The answer to part c) of Problem 2—which is 181—was more difficult to obtain because the summands 15 and 14 were quite large and therefore more natural numbers had to be tested.

Problem 3 was solved by most of the pupils. They discovered that there is no natural number which cannot be written as a sum of numbers taken from $S = \{1, 9\}$, since any natural number n is the sum of n summands equal to 1. This settles part a) of Problem 3. In case b) both elements of $S = \{4, 20\}$ are divisible by 4; thus any number of the form $a \cdot 4 + b \cdot 20$ with $a, b \in \mathbf{N} \cup \{0\}$ must be divisible by 4. There are infinitely many natural numbers not divisible by 4; therefore there is no greatest natural number which cannot be written as a sum of numbers from $S = \{4, 20\}$.

To *Problems 4* and *5* many pupils could provide only partial answers. They found out that there is no greatest natural number n such that $n \neq ap + bq$ for $a, b \in \mathbf{N} \cup \{0\}$ if at least one of the numbers p and q is equal to 1, or if one of the numbers p and q is divisible by the other. Only a few advanced youngsters remarked that there is no greatest number n of the form $n \neq ap + bq$, with $a, b \in \mathbf{N} \cup \{0\}$ whenever the greatest common divisor of p and q is greater than 1.

Problem 6 proved to be too difficult for all pupils. Some of them guessed the correct answer: $g(p, q) = pq - (p + q)$, however they could not prove the validity of this formula. At this stage it seemed worth while to provide some hints:

It was suggested to return to Problem 1 and, instead of looking for numbers n not expressible as sums of elements from $S = \{3, 5\}$, to list all natural numbers of the form $3a + 5b$ for $a, b \in \mathbf{N} \cup \{0\}$. This can be carried out systematically as shown in Table 1:

	3	6	9	12	15	...
5	8	11	14	17	20	...
10	13	16	19	22	25	...
15	18	21	24	27	30	...
20	23	26	29	32	35	...
...	...	...	...	...	...	...

Table 1

The first row of Table 1 consists of the multiples of 3, the second row of the numbers $5 + 3k$ for $k = 0, 1, 2, \ldots$, the third row of the numbers $2 \cdot 5 + 3k$ for $k = 0, 1, 2, \ldots$. In general, the i-th row for $i \geqslant 1$ is the set of the natural numbers of the form $(i - 1) \cdot 5 + 3k$ for $k \geqslant 0$.

Pupils were asked to discover properties of Table 1. This task, related to a specific number pattern, motivated all pupils (also the weakest among them) to carry out further research. They discovered the following:

• All numbers in the first three rows are different from one another.

• The elements of the fourth row are contained in the first row; the elements of the fifth row are contained in the second row, and the elements of the sixth row are in the third row. In general, any element in the i-th row for $i \geqslant 4$ is contained in one of the first three rows of Table 1. Therefore attention can be restricted to the first three rows of the Table.

• Table 1 contains 3, 5, 6 and all natural numbers $n \geqslant 8$. The elements of the sequence 8, 9, 10, 11, 12, 13, ... appear one after the other on the lines parallel to the straight line passing through 10, 11 and 12 (see Table 2).

At this stage Table 2 was constructed. It consists of the first three rows of Table 1 each of which is extended to the left by listing the elements $3-3k$, $5-3k$ and $10-3k$ in the first, second and third row respectively for $k = 1, 2, 3, \ldots$:

...	−12	−9	−6	−3	0	3	6	9	12	15	18	...
...	−7	−4	−1	2	5	8	11	14	17	20	23	...
...	−2	1	4	7	10	13	16	19	22	25	28	...

Table 2

From Table 2 it is clear that the greatest natural number which did not appear in Table 1 is the number in the bottom row of Table 2, preceding the broken line which separates the part of Table 2 corresponding to Table 1 from the rest. Thus, $g(3,5) = 7$.

Parts a), b) and c) of Problem 2 were revisited in the same way: Tables, similar to Table 2 were constructed for the numbers $2a + 7b$, $6a + 5b$ and $15a + 14b$ for $a, b \in \mathbf{Z}$. This led to the guess: $g(p,q) = pq - (p+q)$ for any two coprime natural numbers p and q, greater than 1.

The proof of this conjecture in the general case was carried out later with a selected group of well advanced pupils. For this the following table has been constructed:

...	$-p$	0	p	$2p$	...
...	$q-p$	q	$q+p$	$q+2p$	...
...	$2q-p$	$2q$	$2q+p$	$2q+2p$	...
...	$3q-p$	$3q$	$3q+p$	$3q+2p$	...
...	...	...	...	...	...
...	$(p-1)q-p$	$(p-1)q$	$(p-1)q+p$	$(p-1)q+2p$	...

Table 3

The following properties of Table 3 were verified:

(1) In each (horizontal) row the numbers are monotonously increasing when proceeding from left to the right.

(2) In each (vertical) column the numbers are monotonously increasing when proceeding from the top to the bottom.

(3) Let r_i be the remainder of iq when divided by p for $i = 0, 1, \ldots, p-1$. Then:

(i) all numbers of the i-th row have the remainder r_i when divided by p;

(ii) all integers with remainder r_i when divided by p are contained in the i-th row.

(4) For any two integers i_1, i_2, with $0 \leqslant i_1 < i_2 \leqslant p-1$, the remainders r_{i_1} and r_{i_2} are different. Hence $\{r_0, r_1, \ldots, r_{p-1}\} = \{0, 1, 2, \ldots, p-1\}$.

The above statements imply that Table 3 represents the set $\mathbf{Z}$ of all integers. The greatest natural number not expressible as a sum $aq + bp$ with $a, b \in \mathbf{N} \cup \{0\}$

is in the bottom row, in front of the number $(p-1)q$; that is $g(p,q) = pq-(p+q)$. This solves Problem 6.

REMARK. Problem 6 is a special case of a number-theoretical problem stated by the German mathematician Frobenius (1849–1917): Determine the greatest natural number n, for which the equation $n = \sum_{i=1}^{k} a_i x_i$, where a_i are relatively prime natural numbers greater than 1, has no solution in the set of non-negative integers. For $k = 2$ the problem was solved by Frobenius; he also showed that there are $(a_1-1)(a_2-1)/2$ natural numbers which cannot be written in the form of the above sum. For $k = 3$ the solution was found in the 1970-ies (see [2]). The case $k \geqslant 4$ remains still open.

Part II

Table 1 shows that some natural numbers can be written in more than one way as sums of numbers from the set $\{3,5\}$. This observation led to Problem 7 for advanced pupils aged 16–18:

PROBLEM 7. Find the number of all representations of any natural number n as a sum with summands from the set $\{1,2\}$.

The following approaches to the solution of this problem were observed at work with a group of youngsters [2]:

a) Some pupils tried to establish systematically the total number a_n of representations of n as a sum of the numbers 1 and 2 for small values of n. They obtained the following table:

number n	ways of representing n as a sum	a_n
1	$1=1$	1
2	$2=1+1$ $2=2$	2
3	$3=1+1+1$ $3=2+1$ $3=1+2$	3
4	$4=1+1+1+1$ $4=2+1+1$ $4=1+2+1$ $4=1+1+2$ $4=2+2$	5
...	...	...

Table 4

A glance at Table 4 reveals that there are two kinds of sums for n: sums with the last summand equal to 1 and sums with the last summand equal to 2. There are

a_{n-1} sums of the first kind and a_{n-2} sums of the second for each $n \geqslant 3$. Thus

$$a_n = a_{n-1} + a_{n-2} \quad \text{for} \quad n \geqslant 3, \qquad (1)$$
$$a_1 = 1, \quad a_2 = 2$$

The recursion formula (1) with the initial conditions $a_1 = 1$, $a_2 = 2$ characterizes the famous Fibonacci numbers f_n for $n \geqslant 1$. The Fibonacci numbers, named in honour of Fibonacci of Pisa (1170–1240), are the members of the Fibonacci sequence $f_0, f_1, f_2, \ldots$, where $f_0 = 1$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 2$.

Thus the pupils. following this approach for the solution of Problem 7, concluded that

$$a_n = f_n \quad \text{for all} \quad n \in \mathbf{N}. \qquad (2)$$

b) There were on the other hand youngsters who decided to classify sums for n according to the number of summands equal to 2. It was easy to show the following: There are $\binom{n-i}{i}$ sums for n containing exactly i summands equal to 2, for $0 \leqslant i \leqslant [n/2]$, where $[n/2]$ denotes the greatest integer not exceding $n/2$. Hence

$$a_n = \sum_{i=0}^{[n/2]} \binom{n-i}{i}. \qquad (3)$$

Relationships (2) and (3) together led to the famous formula of the 19-th century French mathematician Lucas:

$$f_n = \sum_{i=0}^{[n/2]} \binom{n-i}{i}. \qquad (4)$$

This formula has been now rediscovered by youngsters—a nice achievement in the classroom.

The pupils were familiar with Pascal's triangle consisting of the binomial coefficients $\binom{n}{k}$ for $n, k \in \mathbf{N} \cup \{0\}$ with $n \geqslant k$. Formula (4) shows that the binomial coefficients in the sums for f_n are situated on the straight lines through $\binom{n}{0}$, parallel to the straight line through $\binom{2}{0}$ and $\binom{1}{1}$:

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\ & & & & \cdots\cdots & & & & \end{array}$$

Pascal's triangle

The above visualization of (4) resulted in attempts to generalize Lucas' formula:

PROBLEM 8. Draw the straight line g_j accross Pascal's triangle, passing through the entries $\binom{j}{0}$ and $\binom{1}{1}$. Cover all entries of Pascal's triangle with straight lines parallel to g_j and determine the sums of the binomial coefficients on each of these parallel lines. These sums form a number sequence S_j. Study the properties of S_j for various values of j.

Finally let us consider the following generalization of Problem 7:

PROBLEM 9. Determine the number t_n of the representations of any natural number n as a sum with summands from the set $\{1, 2, 3\}$.

It can be easily shown, by constructing a table analogous to Table 4, that t_n satisfies the recursion formula

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} \quad \text{for} \quad n \geqslant 3$$

with the initial conditions

$$t_0 = t_1 = 1, \quad t_2 = 2.$$

On the other hand t_n can be expressed as the sum of certain trinomial coefficients, that is numbers of the form $\dfrac{p!}{q!\,r!\,s!}$ with $q + r + s = p$. (These expressions are obtained when one classifies the sums for t_n with summands from $\{1, 2, 3\}$ according to the number of the summands equal to 1, 2 and 3.) It was left as an excercise for the most advanced pupils to establish a formula for t_n in terms of trinomial coefficients.

Part III

It is important to introduce pupils at an early age to work independently, carrying out investigations on various topics. This task is made easier by the fact that young children possess a fair degree of intuition and phantasy enabling them to make interesting observations and suggestions for further steps at work. Here I will concetrate attention on two topics designed for children aged 10–13.

(a) Construction of models for polyhedra (for 10 years old pupils)

The youngsters had at their disposal a large number of congruent, equilateral triangles cut out from a cardboard, and plenty of selotape. They were given the following problem:

PROBLEM 10. Construct as many different shapes as you can by sticking together some of the triangles.

The diversity of the shapes, constructed by the pupils was amazing: they produced convex, concave and starlike bodies of various kinds. After having completed their practical work the pupils were asked to write down their observations about the bodies and to state with explanations which of the shapes have impressed them most. Some of the "mathematically minded" youngsters found the bodies with 4 faces and with 8 faces especially interesting, because as they pointed out "one can turn them in various directions and they still look the same". In other words they discovered two of the Platonic solids: the regular tetrahedron and the octahedron. The activities carried out by the pupils paved the way to the introduction of the three remaining Platonic solids: the icosahedron (also made by congruent equilateral triangles as faces), the cube (which was already well known to the children) and the dodecahedron. There was also opportunity to mention details about Plato, relevant to the history of Mathematics.

Some of the pupils remarked the following: All bodies constructed from triangles as faces had an even number of faces. The age of the pupils did not allow me to provide them with a proof of the general statement:

- There is no polyhedron with an odd number of faces all of which are triangles.

This can be shown at a later stage, by combinatorial arguments as follows: Let P be a polyhedron with triangular faces. Denote by f the number of the faces and by e the number of edges of P. We shall count the number i of the pairs of incident faces and edges. (A pair consisting of a face and of an edge is called *incident* if the edge belongs to the face.) Since each face of P has three edges, it is $i = 3f$. On the other hand, since each edge belongs to two faces, we have $i = 2e$. Thus the equation $3f = 2e$ holds. This implies that $3f$ must be divisible by 2; that is, f must be an even number.

Another problem, involving models of cubes, proved to be useful in developing spatial awareness. To start with, the pupils were acquainted with the notion of a hexomino. A hexomino is a shape in the plane, consisting of six congruent squares with the following properties: 1) Any square of the shape has a common edge with at least one of the remaining squares. 2) Any two squares of the shape have either no point, or a vertex, or an edge in common. The youngsters were asked to solve the following problem:

PROBLEM 11. a) Draw on a cardboard as many different hexaminoes as you can and mark those of your drawings which in your opinion represent the net of a cube.

b) Having done that, cut out all of your hexaminoes and fold them along edges common to pair of squares, trying to form cubes from the folded shapes. Find out whether you have solved correctly part a) of the problem.

(b) Application of figurate numbers in arithmetic and in algebra

In ancient Greece certain types of natural numbers were represented by patterns of dots, in the shape of geometric figures. For example the numbers equal to the products $1 \cdot 1$, $2 \cdot 2$, $3 \cdot 3$, $4 \cdot 4$, ... were represented by dots in square arrays:

Figure 1

This representation justifies the name "square number" for a number of the form n^2, with $n \in \mathbf{N}$.

Numbers equal to the sums 1, 1+2, 1+2+3, 1+2+3+4, ... , were represented by dots forming equilateral triangles:

Figure 2

Hence the number equal to the sums $1+2+\cdots+n$ was called "triangular number".

Pupils aged 10 were presented with Figure 1. It was pointed out to them that in any of the squares the dots in each row can be connected by a straight line segment. In the first, second, third and fourth square each segment contains 1, 2, 3 and 4 dots respectively. Thus the total number of dots in each of the squares is expressible as a sum of the form 1, $2+2$, $3+3+3$, $4+4+4+4$, ...

After that the following problem was stated:

Problem 12. a) Extend Figure 1 by drawing square arrays with $5 \cdot 5$, $6 \cdot 6$, $7 \cdot 7$ dots. b) Draw a square array consisting of 25 dots and suggest various ways of partitioning the set of all dots enabling to express the number 25 as a sum of natural numbers in different ways.

By joint efforts the following solutions to problem 12b) were obtained:

$25 = 5 \cdot 5$

$25 = 1+2+3+4+5+4+3+2+1$

$25 = 1+3+5+7+9$

Figure 3

Finally, the pupils were given:

Problem 13. a) Make drawings, similar to those in Figure 3, for various square numbers. Use them to express the total number of dots as sums of different types.

b) Express the number 100 as a sum of natural numbers; in how many ways can you do this, by following the methods applied in part a) of this problem?

The solution of Problem 13 by the pupils was followed up by a discussion. It became clear, that the pupils were aware of the validity of the formulae:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \tag{5}$$

and

$$n^2 = 1 + 2 + 3 + \cdots + (n - 1) + n + (n - 1) + \cdots + 3 + 2 + 1. \tag{6}$$

The formal proof of the above formulae—by using mathematical induction had to be abandoned at this stage; nevertheless the children understood their meanings.

The ancient Greeks used the number patterns for triangular and square numbers to prove the relations:

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \tag{7}$$

and

$$t_n + t_{n+1} = (n + 1)^2, \tag{8}$$

where t_n denotes the n-th triangular number. Their method was explained to the pupils with the help of drawings for small values of n. Figure 4 presents proofs of relations (7) and (8) for $n = 4$:

$$2 \cdot (1 + 2 + 3 + 4) = 5 \cdot 4$$

i.e.

$$1 + 2 + 3 + 4 = \frac{4 \cdot 5}{2}$$

$$\frac{4 \cdot 5}{2} + \frac{3 \cdot 4}{2} = 4^2$$

Figure 4

REMARK. Triangular numbers have more properties which can be illustrated graphically; one of them, known already to Plutarch (about 45–120 AD) is the following:

- n is a triangular number if and only if $8n + 1$ is a square.

12–13 years old pupils were asked to solve the following problem:

PROBLEM 14. In Figure 5a) there are $t_n = 1 + 2 + \cdots + n$ dots arranged in the shape of a right-angled triangle Δ_n. Eight copies of Δ_n, together with an additional dot, can be assembled to form a square array Q. Determine the square number, represented by the number of the dots in Q.

(a) (b)

Figure 5

b) A square array Q is constructed from k^2 dots. Show that if $k^2 > 4$, then Q can be decomposed into eight right-angled triangles and a square containing a single dot.

c) Deduce from a) and b) that the following assertion holds: A natural number x is a triangular number if and only if $8x + 1$ is the square of a natural number.

A solution of Problem 14 is shown for $n = 4$, in Figure 5.

In the general case the following formula can be proved:

$$8t_n + 1 = (2n + 1)^2.$$

It can be easily verified by the use of algebra that indeed

$$8 \cdot \frac{n(n+1)}{2} + 1 = (2n + 1)^2.$$

Figurate numbers can be used on various occasions with more advanced pupils. For example, square arrays of dots can be applied to find infinitely many—although not all—solutions of "Pythagorean equation"

$$a^2 + b^2 = c^2$$

in natural numbers a, b and c. This is done in the following way: Figure 6 shows that

$$n^2 + (2n + 1) = (n + 1)^2 \tag{9}$$

for any integer $n \geqslant 1$. Namely, the whole square in Figure 6 contains $(n+1)^2$ dots, while the smaller square inside the pattern consists of n^2 dots. In the "corridor" between the two squares there are $2n + 1$ dots.

Figure 6

In order to obtain solutions of Pythagorean equation in natural numbers n must be chosen such that $2n+1$ is a square number. Since $2n+1$ is odd, it has to be the square of an odd number, say $2n+1=(2k+1)^2$. This implies that

$$n = 2k^2 + 2k.$$

Thus (9) can be rewritten in the form

$$(2k^2+2k)^2 + (2k+1)^2 = (2k^2+2k+1)^2.$$

The above identity holds for all natural numbers k.

REFERENCES

[**1**] Z. Mihalewicz and D. B. Fogel: *How to solve it: Modern heuristics*, Springer, Berlin–Heidelberg, 2000.

[**2**] K.-H. Sänger: *Projektorientierter Mathematikunterricht Anregungen zum entdeckenden Lernen und Problemlösen*, Ph. D. Thesis, Erlangen.

Mathematisches Institut, Universität Erlangen-Nürnberg, Erlangen, Germany