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A GENERALIZATION OF PTOLEMY'S THEOREM

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Abstract. In this paper, we introduce a novel generalization of the classic Ptolemy's theorem, focusing on its triangle version. We explore this generalization's implications and provide several applications that illustrate its utility in geometric problem-solving.

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1. Introduction

Ptolemy's theorem (see [2–4, 6–8, 9, 14, 15]) is a well-known result in plane geometry, particularly concerning cyclic quadrilaterals. The theorem is stated as follows.

THEOREM 1. [Quadrilateral Version of Ptolemy's Theorem] *Let $ABCD$ be a convex quadrilateral inscribed in a circle. Then,*

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

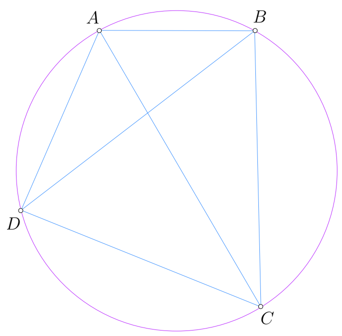

Fig. 1. Illustration of the quadrilateral version of Ptolemy's theorem

This theorem is a powerful tool for solving elementary geometric problems, especially those involving cyclic quadrilaterals, circles, and properties related to angles and distances. The theorem is named after Claudius Ptolemy, a prominent mathematician, astronomer, and geographer who lived around the 2nd century CE. In mathematics, particularly geometry, Ptolemy made significant contributions. He developed various trigonometric theorems and methods for calculating the positions of celestial bodies, including his theorem concerning cyclic quadrilaterals. This theorem has become an essential tool in elementary geometry, especially in problems involving circles.

Ptolemy's theorem has several extensions and generalizations, such as becoming an inequality for quadrilaterals, extending to polygons, Casey's theorem (see [4]), and even generalizing to three-dimensional space.

Ptolemy's theorem can also be expressed in a different form for triangles, as follows.

Theorem 2. [Triangle Version of Ptolemy's Theorem] *Let ABC be a triangle inscribed in a circle ω. Let P be a point on the arc BC that does not contain A. Then,*

$$a \cdot PA = b \cdot PB + c \cdot PC,$$

where a, b, and c are the lengths of the sides BC, CA, and AB of the triangle ABC, respectively.

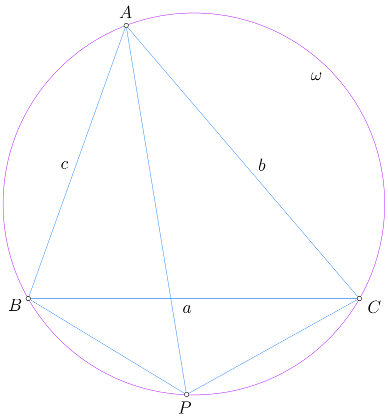

Fig. 2. Illustration of the triangle version of Ptolemy's theorem

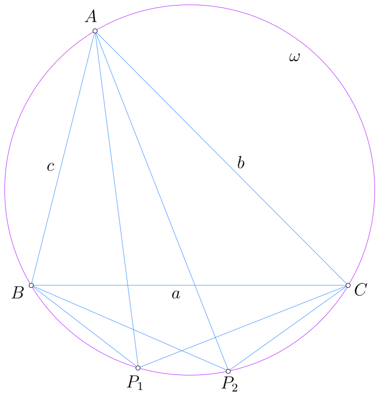

Fig. 3. Illustration of the generalization of the triangle version of Ptolemy's theorem

The first author of this article has previously provided a generalization of Ptolemy's Theorem in [10] and applied it to extend Pythagorean Theorem (see [11]). Additionally, the author has proven a generalized version of Pythagorean Theorem using Ptolemy's Theorem (see [12]).

In this paper, we introduce another generalization of Theorem 2 as follows

Theorem 3. [Generalization of the triangle version of Ptolemy's Theorem] *Let ABC be a triangle inscribed in a circle ω. Let P_1 and P_2 be any points on the arc BC that does not contain A. Then,*

$$a\sqrt{P_1A \cdot P_2A} \geq b\sqrt{P_1B \cdot P_2B} + c\sqrt{P_1C \cdot P_2C},$$

where a, b, and c are the lengths of the sides BC, CA, and AB of the triangle ABC, respectively. Equality holds if and only if $P_1 = P_2$.

Clearly, when $P_1 = P_2$, Theorem 3 reduces to Theorem 2, thus making Theorem 3 a generalization of Theorem 2.

We will provide a proof of Theorem 3 in the following section, along with several applications of this generalization in the subsequent sections.

2. Proof of Theorem 3

Let P be the midpoint of the arc P_1P_2 that does not contain the triangle vertex A on ω. It is evident that $PP_1 = PP_2$, and we denote $PP_1 = PP_2 = k$. Let D be the intersection of AP and P_1P_2. Since P is the midpoint of the arc P_1P_2, it follows that AP bisects $\angle P_1AP_2$. Combined with the equality of the inscribed angles $\angle AP_1P_2 = \angle APP_2$, we find that triangles AP_1D and APP_2 are similar (by angle-angle similarity). As a result, we have

$$AP_1 \cdot AP_2 = AD \cdot AP. \tag{1}$$

Moreover, since $PP_1 = PP_2$, we also get $\angle PP_1D = \angle PP_2P_1 = \angle PAP_1$. Thus, triangles PP_1D and PAP_1 are similar (by angle-angle similarity), leading to

$$k^2 = PP_1^2 = PD \cdot PA. \tag{2}$$

Combining (1) and (2), we obtain

$$AP_1 \cdot AP_2 + k^2 = PA(PD + AD) = PA^2. \tag{3}$$

Similarly, we also get

$$BP_1 \cdot BP_2 + k^2 = PB^2 \quad \text{and} \quad CP_1 \cdot CP_2 + k^2 = PC^2. \tag{4}$$

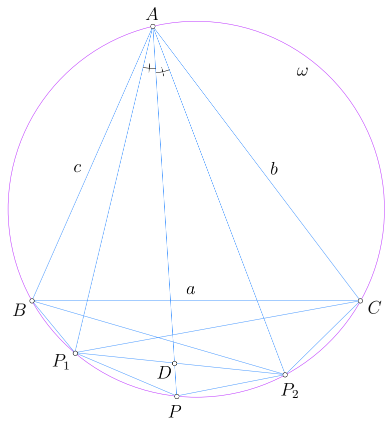

Fig. 4. Illustration for the proof of Theorem 3

It is clear that when P_1 and P_2 lie on the arc BC that does not contain A on ω, the midpoint P also lies on the same arc. Applying Ptolemy's theorem, we have

$$a \cdot PA = b \cdot PB + c \cdot PC. \tag{5}$$

Using (3)–(5), we derive

$$a\sqrt{P_1A \cdot P_2A + k^2} = b\sqrt{P_1B \cdot P_2B + k^2} + c\sqrt{P_1C \cdot P_2C + k^2}. \tag{6}$$

By squaring both sides of (6) and simplifying, we obtain

$$\begin{aligned} a^2(P_1A \cdot P_2A + k^2) &= b^2(P_1B \cdot P_2B + k^2) + c^2(P_1C \cdot P_2C + k^2) \\ &\quad + 2bc\sqrt{(P_1B \cdot P_2B + k^2)(P_1C \cdot P_2C + k^2)}. \end{aligned} \tag{7}$$

Using the Cauchy-Schwarz inequality, we have

$$(P_1B \cdot P_2B + k^2)(P_1C \cdot P_2C + k^2) \geq \left(\sqrt{P_1B \cdot P_2B \cdot P_1C \cdot P_2C} + k^2\right)^2. \tag{8}$$

Thus, combining (7) and (8), we derive

$$\begin{aligned} &a^2(P_1A \cdot P_2A + k^2) \\ &\geq b^2(P_1B \cdot P_2B + k^2) + c^2(P_1C \cdot P_2C + k^2) + 2bc\left(\sqrt{P_1B \cdot P_2B \cdot P_1C \cdot P_2C} + k^2\right) \\ &= \left(b\sqrt{P_1B \cdot P_2B} + c\sqrt{P_1C \cdot P_2C}\right)^2 + (b+c)^2k^2 \end{aligned} \tag{9}$$

Using triangle inequality, we have $b + c > a$, therefore $(b + c)^2 > a^2$ so that

$$(b+c)^2k^2 > a^2k^2 \text{ (if } k > 0) \text{ and } (b+c)^2k^2 = a^2k^2 \text{ (if } k = 0). \tag{10}$$

From (9) and (10), it follows that

$$a^2(P_1A \cdot P_2A) \geq \left(b\sqrt{P_1B \cdot P_2B} + c\sqrt{P_1C \cdot P_2C}\right)^2, \tag{11}$$

or equivalently,

$$a\sqrt{P_1A \cdot P_2A} \geq b\sqrt{P_1B \cdot P_2B} + c\sqrt{P_1C \cdot P_2C}.$$

It is easy to see that equality holds when $k = 0$, or $P_1 = P_2$. This concludes the proof.

3. Some applications

In this section, we will explore several interesting corollaries of Theorem 3.

It is well known that Ptolemy's theorem generalizes van Schooten's theorem (see [1, 13, 15]). Therefore, by applying Theorem 3 to an equilateral triangle ABC, we obtain a generalization of van Schooten's theorem as follows.

Corollary 1. [Generalization of van Schooten's Theorem] *Let ABC be an equilateral triangle inscribed in a circle ω. Let P_1 and P_2 be points on the minor arc BC of ω. Then,*

$$\sqrt{P_1A \cdot P_2A} \geq \sqrt{P_1B \cdot P_2B} + \sqrt{P_1C \cdot P_2C}.$$

Next, we consider another application, which is also a corollary of Theorem 3.

Corollary 2. *Let ABC be a triangle inscribed in a circle ω. Let P_1 and P_2 be any points on the arc BC that does not contain A. Then,*

$$a\sqrt{S_a} \geq b\sqrt{S_b} + c\sqrt{S_c},$$

where a, b, and c are the lengths of the sides BC, CA, and AB of triangle ABC, and S_a, S_b, S_c denote the areas of triangles AP_1P_2, BP_1P_2, and CP_1P_2, respectively. Equality holds if and only if $P_1 = P_2$.

Proof. We note that the inscribed angles are equal:

$$\angle P_1AP_2 = \angle P_1BP_2 = \angle P_1CP_2.$$

Let these angles be denoted as α. Then, by Theorem 3, we have

$$a\sqrt{P_1A \cdot P_2A} \geq b\sqrt{P_1B \cdot P_2B} + c\sqrt{P_1C \cdot P_2C}. \tag{12}$$

Since $\sin\alpha \geq 0$, multiplying both sides of 12 by $\sqrt{\sin\alpha}$ yields

$$a\sqrt{P_1A \cdot P_2A \sin\alpha} \geq b\sqrt{P_1B \cdot P_2B \sin\alpha} + c\sqrt{P_1C \cdot P_2C \sin\alpha}, \tag{13}$$

or equivalently,

$$a\sqrt{S_a} \geq b\sqrt{S_b} + c\sqrt{S_c}. \quad \blacksquare$$

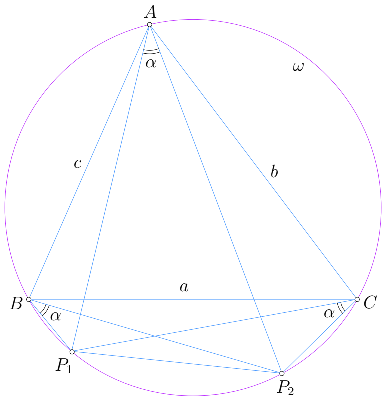

Fig. 5. Illustration for the proof of Corollary 2

In Corollary 2, if we assume that ABC is equilateral, we obtain the following result:

Corollary 3. *Let ABC be an equilateral triangle inscribed in a circle ω. Let P_1 and P_2 be any points on the arc BC that does not contain A. Then,*

$$\sqrt{S_a} \geq \sqrt{S_b} + \sqrt{S_c},$$

where S_a, S_b, and S_c denote the areas of triangles AP_1P_2, BP_1P_2, and CP_1P_2, respectively. Equality holds if and only if $P_1 = P_2$.

We derive two more simple corollaries from Corollaries 2 and 3 as follows

Corollary 4. *Let ABC be a triangle inscribed in a circle ω. Let ℓ be any line that either intersects or is tangent to the arc BC of ω that does not contain A. Then,*

$$a\sqrt{d_a} \geq b\sqrt{d_b} + c\sqrt{d_c},$$

where a, b, and c are the lengths of the sides BC, CA, and AB of triangle ABC, and d_a, d_b, d_c denote the perpendicular distances from points A, B, and C to the line ℓ, respectively. Equality holds if and only if ℓ is tangent to the minor arc BC of ω.

Proof. If ℓ intersects the arc BC not containing A at two distinct points P_1 and P_2, let S_a, S_b, and S_c represent the areas of triangles AP_1P_2, BP_1P_2, and CP_1P_2, respectively. Then,

$$S_a = \frac{1}{2}P_1P_2 \cdot d_a, \; S_b = \frac{1}{2}P_1P_2 \cdot d_b, \; S_c = \frac{1}{2}P_1P_2 \cdot d_c. \tag{14}$$

By Corollary 2, we have

$$a\sqrt{S_a} > b\sqrt{S_b} + c\sqrt{S_c}. \tag{15}$$

(Note that this inequality is strict because P_1 and P_2 are distinct points). Thus, from (14) and (15) (noting that $P_1P_2 > 0$), we deduce

$$a\sqrt{d_a} > b\sqrt{d_b} + c\sqrt{d_c}.$$

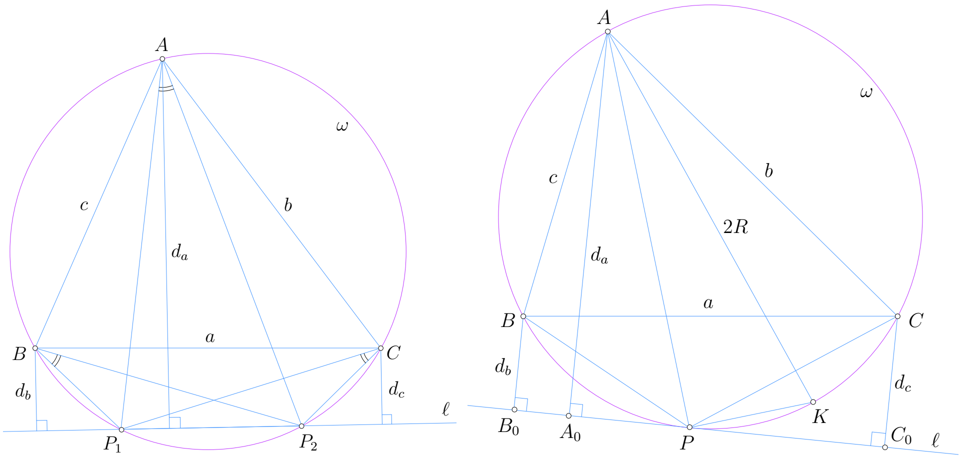

Fig. 6. Illustrations for the proof of Corollary 4

Now, if ℓ is tangent to the arc BC that does not contain A at a point P, let AK be the diameter of the circle ω, and let R denote the radius of ω so that $AK = 2R$. Let A_0, B_0, and C_0 be the perpendicular projections of A, B, and C onto ℓ. Since ℓ is tangent to the minor arc BC of ω at P, we have $\angle APA_0 = \angle AKP$. Furthermore, since APK is a right triangle at P (as AK is the diameter of ω), triangles AA_0P and APK are similar (by the angle-angle criterion). It follows that $PA^2 = AK \cdot AA_0 = 2R \cdot d_a$, i.e.

$$PA = \sqrt{2R \cdot d_a}. \tag{16}$$

Similarly, we have

$$PB = \sqrt{2R \cdot d_b}, \quad \text{and} \quad PC = \sqrt{2R \cdot d_c}. \tag{17}$$

Now, by applying the triangle form of Ptolemy's theorem to triangle ABC with P lying on the arc BC not containing A, we obtain

$$aPA = bPB + cPC. \tag{18}$$

From (16)–(18), we conclude that

$$a\sqrt{2R \cdot d_a} = b\sqrt{2R \cdot d_b} + c\sqrt{2R \cdot d_c},$$

or equivalently,

$$a\sqrt{d_a} = b\sqrt{d_b} + c\sqrt{d_c}. \quad \blacksquare$$

From here, we arrive at the following corollary when ABC is equilateral in Corollary 4:

Corollary 5. *Let ABC be an equilateral triangle inscribed in a circle ω. Let ℓ be any line that intersects or is tangent to the minor arc BC of ω. Then,*

$$\sqrt{d_a} \geq \sqrt{d_b} + \sqrt{d_c},$$

where d_a, d_b, and d_c denote the distances from points A, B, and C to the line ℓ, respectively. Equality holds if and only if ℓ is tangent to the minor arc BC of ω.

4. Conclusion

In this paper, we have introduced a new generalization of Ptolemy's theorem, extending its classical form to a triangle version. This generalization not only encompasses the original theorem as a special case but also provides a broader framework for exploring relationships between points and distances in cyclic triangles. Through a series of corollaries, we demonstrated how this result can be applied to derive inequalities involving areas and distances, thereby offering new insights into geometric problem–solving.

The results presented here deepen our understanding of the connections between classical theorems in geometry, such as Ptolemy's theorem and van Schooten's theorem, showing their interdependence and the power of generalization. We also highlighted the practical applications of this generalized theorem, which can serve as a valuable tool for tackling challenging problems in advanced Euclidean geometry.

Future research could explore further extensions of this generalization to higher-dimensional spaces or other geometric configurations, such as polygons inscribed in circles. Additionally, the techniques used in this paper may inspire similar generalizations in other branches of geometry, opening up new avenues for study and exploration.

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