THE TEACHING OF MATHEMATICS
2024, Vol. XXVII, No. 2, pp. 59–78
DOI: 10.57016/TM-UZPZ2356

ANGLE TRISECTION WITH ORIGAMI AND PROVING ITS CORRECTNESS USING GRÖBNER BASIS

Dara Milojković and Vesna Marinković

Abstract. In this paper, trisection of an angle performed by origami is explained in detail, as well as the correctness of obtained construction. Two different correctness conjectures, one based on trigonometry identities for triple angle and another, based on triangle congruence are formulated. All geometric constraints appearing in premises and conclusion of the conjecture are formulated as polynomials over the set of appropriately chosen variables, and the correctness conjectures are proved using Gröbner basis method. For performing calculations over polynomials obtained, the computer tool Singular is used.

MathEduc Subject Classification: G44, H24

AMS Subject Classification: 97G40, 97H20

Key words and phrases: Origami constructions; angle trisection; Gröbner basis method; Singular.

1. Introduction

Geometric construction problems are standard mathematical problems in high school education. Geometric constructions are mainly done using straightedge[1] and compass (abbr. SC constructions). This method of solving geometric construction dates back to the times of ancient Greece and is considered a cornerstone of human intellectual reach. Over the past decades, several tools for automated solving SC constructions have been developed [10, 11, 15], however this domain still remains challenging for automation.

Origami (ori – folding, kami – paper) is a traditional Japanese art of paper folding, used to teach children math, as well as to improve their creativity. It turns out that it can be used as an alternative tool for performing geometric constructions. During the centuries, origami found its practical use in the field of solving geometric constructions and today there exist computer programs and tools that can perform origami construction and reason about them [7]. Moreover, with origami one can solve some problems that are proved unsolvable using straightedge and compass, for example doubling the cube or angle trisection.

The main task of geometric construction problem is to construct (using available tools) a geometric figure which satisfies given set of constraints [12]. However, this is only one part of the solution, since a proof of correctness of obtained construction is, also, needed. Namely, proving construction correct is of extreme

[1]By straightedge we consider ruler with no marks.

importance in the context of mathematical education, since for a student it is essential to understand why a performed construction is correct. One of the most widely used methods for automatically proving geometric statements are algebraic methods which work as follows: the properties of a figure are translated into a set of polynomials over coordinates of points and it should be proved that the polynomials describing the conclusion follow from the polynomials specifying the premises of the statement. The most successful algebraic methods are Wu's method [18] and Gröbner basis method [5]. Most computer algebra systems, such as Mathematica, Maple, etc. as a part of standard library have a function for calculating Gröbner basis of an ideal. One such system, that is free and open-source, is Singular [4].

In this paper we will focus on proving correctness of constructions obtained by origami using Gröbner basis method. We will consider the problem of angle trisection and prove the correctness of presented construction in two different ways: using trigonometric identities and the fact that the tangent function is injective on $(0, \pi/2)$ and, alternatively, using congruence of triangles.

In Section 2 geometry constructions performed by origami are described, and the basic principles of Gröbner basis method for theorem proving are given. In Section 3 the problem of angle trisection is discussed and the construction and the corresponding correctness conjecture are specified. In Section 4 the implementation of correctness proof in Singular is described, while in Section 5 the final conclusions are drawn and plans for future work are given.

2. Background

In this section we describe the basic concepts of geometry constructions performed by origami. We list Huzita's origami axioms and formulate them first in logical and then in algebraic form. We also recall Gröbner basis method that can be used for proving geometric theorems.

2.1. Origami geometry construction

Origami geometry construction usually begins with uncut square piece of paper where only folding is allowed[2]. Folding is a simple operation which creates fold lines along the paper. To perform a folding step it is necessary to identify reference points or lines which will be brought or passed through, thus creating a fold line. A sequence of such folding steps creates new reference points and/or lines and the process is repeated until the final model is achieved. Thus, every folding step is a precisely defined combination of paper elements which include points, edges, fold lines and their intersections (see Figure 1).

At the beginning of the 1970s, a group of origami researchers began to systematically enumerate various possible folding combinations and they studied what relationships between objects can be attained by folding [9]. The first research on this subject was conducted by Humiaki Huzita; he described the set of six rules of

[2]For some origami models construction begins with rectangular piece of paper and/or cutting of paper can be allowed.

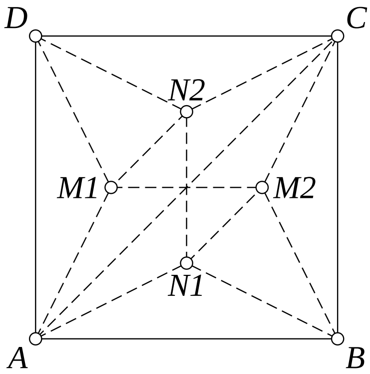


Fig. 1. Origami crane model with fold lines. Solid lines represent paper edges, while dashed lines represent fold lines. Points A, B, C and D are reference points while N_1, N_2, M_1 and M_2 are points of intersection of fold lines.

paper folding using existing points, lines and fold lines. These six rules are well known as Huzita's axioms (HA)[3].

Every step in origami construction corresponds to application of Huzita's axiom to existing points and lines. Let us denote the set of points with **P** and the set of lines with **L**. Each of Huzita's axioms can be specified as a formula in the language of first-order logic, formulated in prenex normal form:

$$\psi = \mathcal{Q}_1 x_1 \cdots \mathcal{Q}_n x_n \ \phi(x_1, \ldots, x_n)$$

where:

- $\mathcal{Q}_i \in \{\forall, \exists\}$, $i \in [1, n]$,
- x_i is a variable, $i \in [1, n]$, that takes values from $\mathbf{P} \cup \mathbf{L}$,
- $\phi(x_1, \ldots, x_n)$ is a formula with free occurrences of variables $x_1, \ldots, x_n$. Formula ϕ can be an atomic formula, a conjunction of atomic formulae, a disjunction of atomic formulae, or a negation of atomic formula.

Formula ϕ does not use any functional symbols. Predicates over which ϕ is formulated are:

- $onLine(X, l)$ – point X is on line l,
- $symmetric(X, Y, l)$ – points X and Y are symmetric with respect to line l,
- $equidistant(X, l, m)$ – point X is equidistant to lines l and m,
- $perp(l, m)$ – lines l and m are perpendicular.

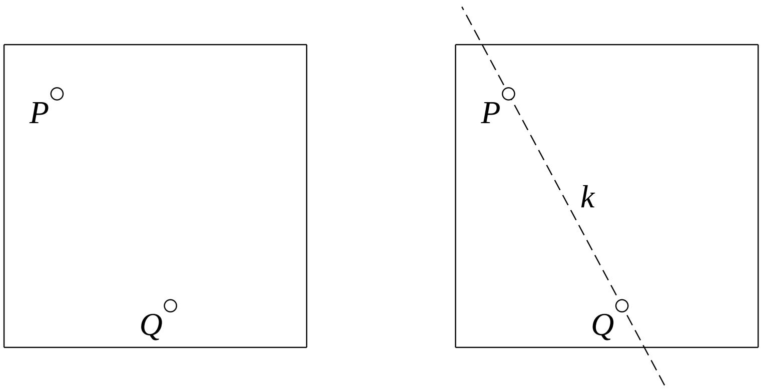
Let us now specify Huzita's axioms and formulate them in the language of first-order logic.

[3] There is also a seventh axiom which was discovered, independently, by Jacques Justin and Koshiro Hatori [9].

HA1. Given two points P and Q, one can construct a fold line k that passes through P and Q (Figure 2).

By this axiom the existence of a line k that contains two points P and Q is asserted. It can be formulated in prenex normal form in the following way:

$$\forall P, Q \in \mathbf{P} \quad \exists k \in \mathbf{L} \quad onLine(P, k) \ \wedge \ onLine(Q, k)$$



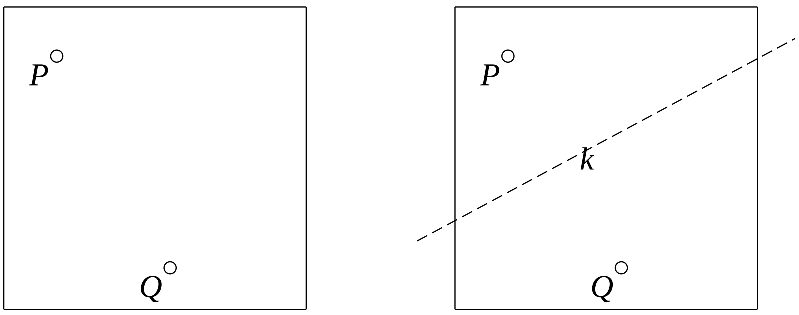
(a) Before folding (b) After folding

Fig. 2. Illustration of application of **HA1** to points P and Q. The fold line k through P and Q is constructed.

HA2. Given two points P and Q, one can construct a fold line k that brings P onto Q (Figure 3).

The second Huzita's axiom claims the existence of a line k such that points P and Q are symmetric with respect to line k and it can be specified as follows:

$$\forall P, Q \in \mathbf{P} \quad \exists k \in \mathbf{L} \quad symmetric(P, Q, k)$$



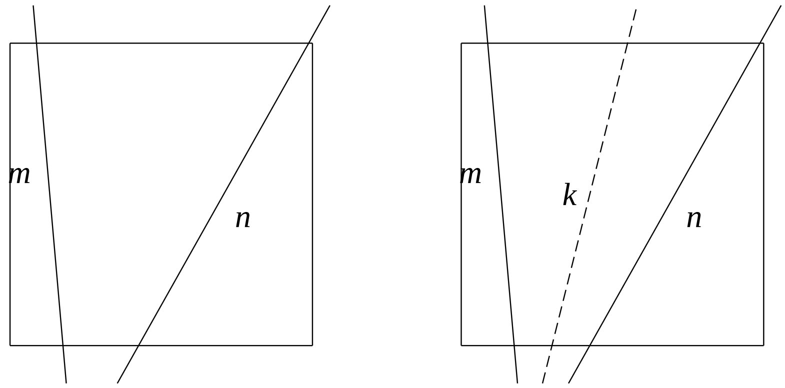
(a) Before folding (b) After folding

Fig. 3. Illustration of application of **HA2** to points P and Q. The fold line k that brings P onto Q is constructed.

HA3. Given two lines m and n, one can construct a fold line k that superposes m and n (Figure 4).

By this axiom the existence of a line k such that each point of the line k is equidistant to lines m and n is claimed. It can be described by the following formula:

$$\forall m, n \in \mathbf{L} \quad \exists k \in \mathbf{L} \quad \forall P \in \mathbf{P} \quad \neg onLine(P, k) \ \vee \ equidistant(P, m, n)$$



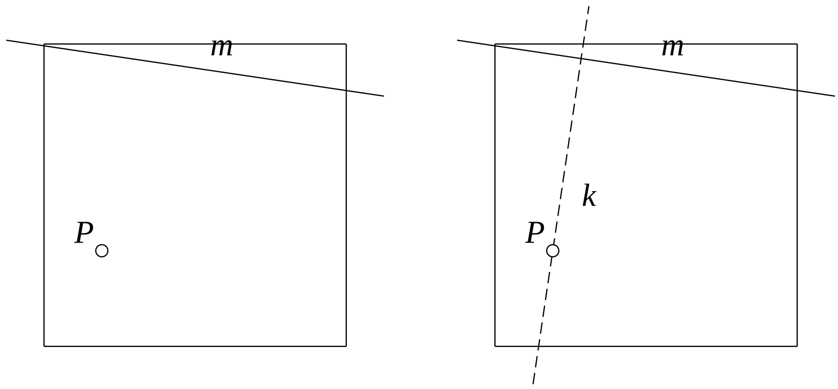
(a) Before folding (b) After folding

Fig. 4. Illustration of application of **HA3** to lines m and n.
The fold line k that superposes m and n is constructed.

HA4. Given a point P and a line m, one can construct a fold line k that passes through P and is perpendicular to m (Figure 5).

This axiom can be specified as follows:

$$\forall P \in \mathbf{P} \quad \forall m \in \mathbf{L} \quad \exists k \in \mathbf{L} \quad onLine(P, k) \ \wedge \ perp(m, k)$$



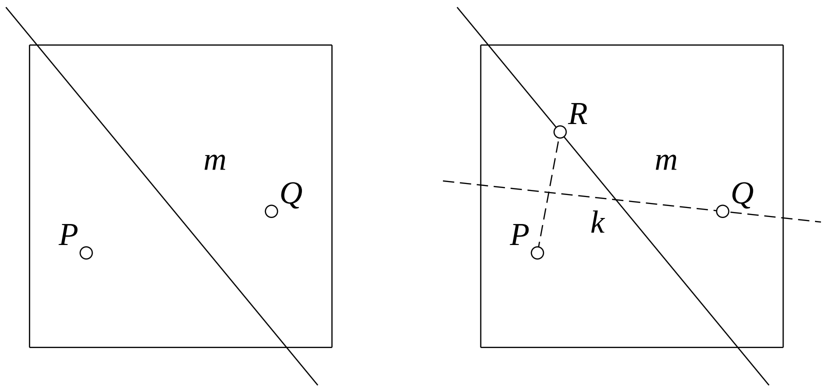
(a) Before folding (b) After folding

Fig. 5. Illustration of application of **HA4** to point P and line m.
The fold line k that passes through P and is perpendicular to m is constructed.

HA5. Given two points P and Q and a line m, one can construct a fold line k that passes through Q and that superposes P and m (Figure 6).

In other words the existence of a line k such that there is a point R that is symmetric to P with respect to line k and m passes through R is asserted.

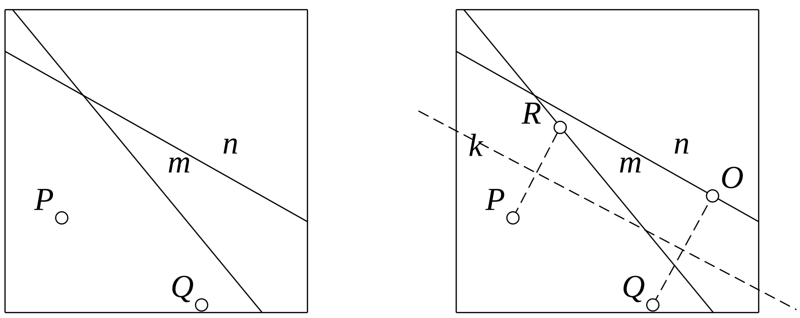
$$\forall P, Q \in \mathbf{P} \quad \forall m \in \mathbf{L} \quad \exists k \in \mathbf{L} \quad \exists R \in \mathbf{P}$$
$$onLine(Q,k) \ \wedge \ symmetric(P,R,k) \ \wedge \ onLine(R,m)$$



(a) Before folding (b) After folding

Fig. 6. Illustration of application of **HA5** to points P and Q and line m. The fold line k that passes through Q and that superposes P and m is constructed.

HA6. Given two points P and Q and two lines m and n, one can construct a fold line k that superposes P and m, and Q and n, simultaneously (Figure 7).



(a) Before folding (b) After folding

Fig. 7. Illustration of application of **HA6** to points P and Q and lines m and n. The fold line k that superposes P and m, and Q and n is constructed.

The last Huzita's axiom asserts the existence of a line k such that there is a point R that belongs to m and is symmetric to P with respect to k, and a point O

that belongs to n and is symmetric to Q with respect to k.

$$\forall P, Q \in \mathbf{P} \quad \forall m, n \in \mathbf{L} \quad \exists k \in \mathbf{L} \quad \exists R, O \in \mathbf{P}$$
$$\mathit{symmetric}(P, R, k) \;\wedge\; \mathit{onLine}(R, m) \;\wedge\; \mathit{symmetric}(Q, O, k) \;\wedge\; \mathit{onLine}(O, n)$$

Once the origami axioms are formulated as formulae in prenex normal form, they can be further translated into an algebraic form. For this reason we introduce a function $\mathtt{m}$ on the set $\mathbf{P} \cup \mathbf{L}$ that maps point $P \in \mathbf{P}$ to a pair of its coordinates (x_P, y_P), and line $l \in \mathbf{L}$ to a triple of its coefficients (a, b, c): these are coefficients of x, y and free term in implicit line equation $ax + by + c = 0$. Without loss of generality it can be assumed that $a^2 + b^2 = 1$ [6][4].

Quantifier-free formulae are then mapped using function $\mathtt{M}$ to a set of polynomial equations with real coefficients in the following way:

- $\mathtt{M}(\bigwedge_{i=1}^{n} \phi_i) = \bigcup_{i=1}^{n} \mathtt{M}(\phi_i)$
- $\mathtt{M}(\bigvee_{i=1}^{n} \phi_i) = \{p_1 \cdot \ldots \cdot p_n = 0\}$, where $p_i = 0 \in \mathtt{M}(\phi_i),\ 1 \leq i \leq n$
- $\mathtt{M}(\neg\phi) = \{\prod_{(p=0)\in\mathtt{M}(\phi)}(p\epsilon_p - 1) = 0\}$, where a new variable ϵ_p is introduced for every formula $p = 0$.
- $\mathtt{M}(\mathit{onLine}(P, k)) = \{ax_P + by_P + c = 0\}$, where $\mathtt{m}(P) = (x_P, y_P)$ and $\mathtt{m}(k) = (a, b, c)$.
- $\mathtt{M}(\mathit{perp}(k, l)) = \{a_k a_l + b_k b_l = 0\}$, where $\mathtt{m}(k) = (a_k, b_k, c_k)$ and $\mathtt{m}(l) = (a_l, b_l, c_l)$.
- $\mathtt{M}(\mathit{midpoint}(R, P, Q)) = \{2x_R - x_P - x_Q = 0, 2y_R - y_P - y_Q = 0\}$ where $\mathtt{m}(P) = (x_P, y_P)$, $\mathtt{m}(Q) = (x_Q, y_Q)$, and $\mathtt{m}(R) = (x_R, y_R)$.
- $\mathit{symmetric}(P, Q, k)$ is an abbreviation of formula written in prenex normal form:

  $$\exists l \in \mathbf{L} \quad \mathit{onLine}(P, l) \wedge \mathit{onLine}(Q, l) \wedge \mathit{perp}(k, l) \wedge \mathit{midpointOnLine}(P, Q, k)$$

  where $\mathit{midpointOnLine}(P, Q, k)$ means that the midpoint of a segment PQ is on the line k.
- $\mathtt{M}(\mathit{midpointOnLine}(P, Q, k)) = \{a(x_P + x_Q) + b(y_P + y_Q) + 2c = 0\}$ where $\mathtt{m}(P) = (x_P, y_P)$, $\mathtt{m}(Q) = (x_Q, y_Q)$ and $\mathtt{m}(k) = (a, b, c)$.
- $\mathtt{M}(\mathit{equidistant}(P, k, l)) = \{d(P, k) - d(P, l) = 0\}$ where $d(Q, n)$ denotes the distance between Q and n and is calculated as $d(Q, n) = \frac{|ax_Q + by_Q + c|}{\sqrt{a^2 + b^2}}$ where $\mathtt{m}(Q) = (x_Q, y_Q)$ and $\mathtt{m}(n) = (a, b, c)$.

Using mapping $\mathtt{M}$ each step of origami construction (application of HA) is easily translated into the corresponding set of polynomials.

[4] This way we obtain normalized implicit line equation that simplifies further calculations is Singular. Note that even with this additional constraint, one can specify a single line in two different ways, however this does not influence further calculation.

2.2. Gröbner basis method

Gröbner basis of an ideal $I = \langle f_1, \ldots, f_k \rangle$ is a set of polynomials $G = \{g_1, \ldots, g_l\}$ for which division of an arbitrary polynomial f with the polynomials of G returns 0 if and only if $f \in I$. Every ideal has a Gröbner basis; it can be constructed using Bucherberger's algorithm [2]. In the worst case scenario, Gröbner basis can be doubly exponentially larger than the input set of the polynomials. Despite that, in most of the cases it can be obtained in a reasonable time [1]. One can implement Buchberger's algorithm from scratch, however there are tools that already implement Buchberger's algorithm for calculating Gröbner basis. One such tool is Singular [4].

Gröbner basis method enables a uniform approach to solving different problems in mathematics that can be expressed in terms of a system of multivariate polynomial equations, such as solving system of algebraic equations, integer programming problems, proving geometrical theorems, etc.

Let us illustrate Gröbner basis method with a simple example. "Alice, Bob and Carol together have 14 coins. If Bob has 4 coins more than Alice and Carol together, check if Alice and Carol have a total of 5 coins." If we denote the number of coins that Alice, Bob, and Carol have with A, B, and C respectively, then the premises of the problem can be specified as a system of two equations over three variables – A, B, and C:

$$A + B + C = 14$$
$$B = A + C + 4$$

Alternatively, the system can be rewritten as a system of two polynomial equations:

$$f_1(A, B, C) = A + B + C - 14 = 0$$
$$f_2(A, B, C) = A - B + C + 4 = 0$$

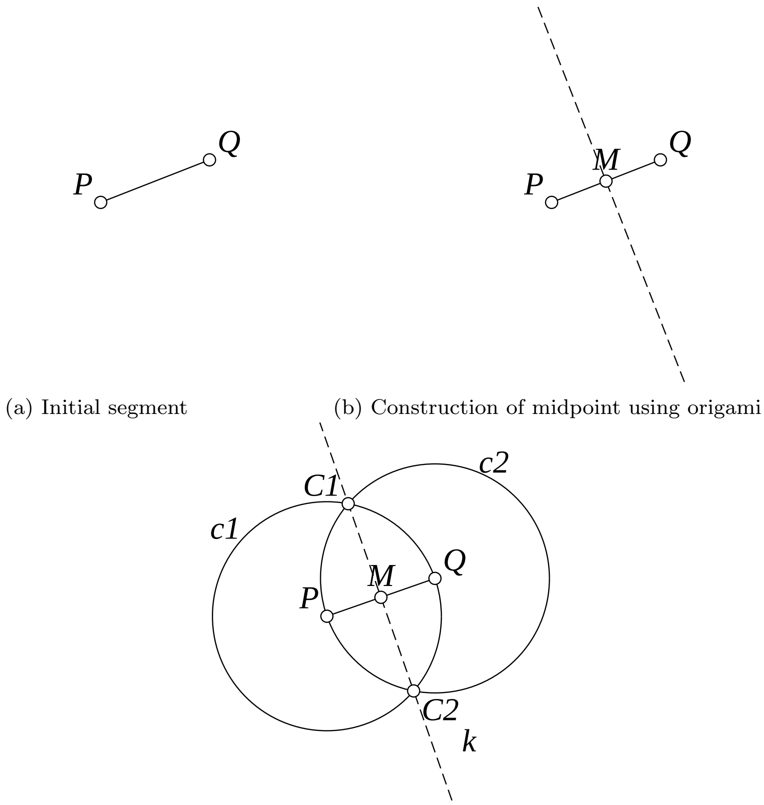
This system has more unknowns than equations, therefore it can't be solved. However, it turns out possible to check whether the conclusion $A + C = 5$ follows from these two equations, i.e. if the conclusion $h(A, B, C) = A + C - 5 = 0$ follows from premises $f_1(A, B, C) = 0$ and $f_2(A, B, C) = 0$. One way to do it is by checking if the polynomial h belongs to the ideal I generated by the polynomials f_1 and f_2. It can be done in the following way: a Gröbner basis G of the ideal I is constructed and the polynomial h is reduced over G – if and only if it yields 0 the polynomial h is in the ideal I.

Gröbner basis method is usually used for solving problems which are considered computationally hard.

In domain of geometry, theorems to be proved are usually given as implications, i.e. one has to prove that the set of premises implies a set of conclusions [13, 14]. This can be achieved using Gröbner basis method by formulating the premises of conjecture as a set of multivariate polynomial equations $f_i = 0,\ i \in [1, k]$, as well as the set of conclusions $h_j = 0, j \in [1, l]$ and proving that each h_j is in ideal $I = \langle f_1, \ldots, f_k \rangle$.

3. Angle trisection problem

SC and origami constructions are similar in many ways. For instance, both approaches support construction of a line through two points, construction of an intersection point of two lines, bisecting an angle, etc. Since SC constructions are much better studied, quite a few solutions of origami constructions have been inspired by corresponding SC constructions. However, some constructions are easier and more intuitive to perform using origami than using straightedge and compass. Let us consider the problem of construction of a midpoint M of a segment PQ (see Figure 8): using origami it suffices to construct the fold line k that brings P to Q and M will be the intersection point of k with PQ. However, using straightedge and compass it is necessary to construct the circle c_1 centered at P through Q as well as the circle c_2 centered at Q passing through P, their intersection points C_1 and C_2, the line k through points C_1 and C_2, and, finally, the intersection M of k with PQ.



(a) Initial segment

(b) Construction of midpoint using origami

(c) Construction of midpoint using SC

Fig. 8. Comparison of origami construction with SC construction of the midpoint M of the segment PQ.

Many construction problems were posed and solved using straightedge and compass back in time of Euclid's, however for some of them the solution could not be reached for a long time. The three such problems are:

- *squaring the circle*, where one has to construct a square with the same area as the given circle;
- *doubling the cube*, where one has to construct a cube with the volume twice as big as the given cube;
- *angle trisection*, where one has to divide a given angle into three equal parts.

Many mathematicians, since the time of ancient Greeks, tried to solve them using straightedge and compass but did not succeed. First known proofs that the cube cannot be duplicated and that the arbitrary angle cannot be divided into three equal parts using straightedge and compass were given by Pierre Wantzel [17], while the unsolvability of problem of squaring the circle using straightedge and compass is a consequence of Lindemann-Weierstrass theorem [16]. However, it shows possible to solve the problems of doubling the cube and angle trisection using origami[5].

The reason for such behaviour lies in algebraic form of these problems. Namely, by using straightedge and compass one can construct lines and circles; their implicit equations are $ax + by + c = 0$ and $x^2 + y^2 - r^2 = 0$, where a, b, c are line coefficients and r is the radius of a circle with the center in point $(0, 0)$. Therefore, using straightedge and compass as the only tools available, one can solve problems that are reducible to linear or quadratic equations [9]. However, problems of cube doubling and angle trisection can be reduced to cubic equations[6], so, using straightedge and compass, one cannot solve any of them. Nevertheless, by using origami one can solve these problems by applying the operation **HA6**[7].

In the rest of this section we will focus on the problem of angle trisection, i.e. the construction of an angle equal to the third of a given angle. We will first present the origami steps used to divide an arbitrary acute angle into three equal parts[8]. Afterwards, we will describe how to formulate the premises and the conclusion of the correctness conjecture of this construction in terms of first-order logic formulae.

3.1. Construction

Trisection of an arbitrary acute angle using origami can be achieved in various ways [9]. One of the construction was proposed by Japanese mathematician Hisashi Abe (see Figure 9). Abe's method for angle trisection consists of the following construction steps:

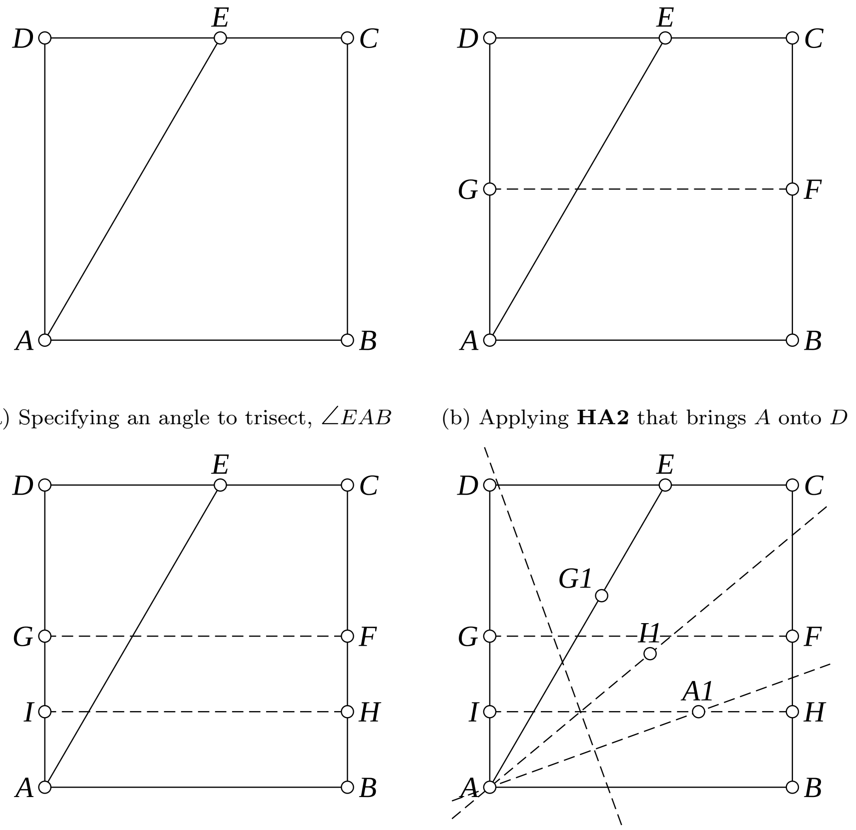
S1 A piece of square paper is defined by four corner points, denoted by A, B, C, and D. These four points form lines AB, BC, CD, and AD. A point E is arbitrarily chosen on edge CD. By constructing the line through points A and E an acute angle $\angle EAB$ to be trisected is formed (see Figure 9(a)).

[5]Note that the neusis construction for angle trisection using tools other than straightedge and compass was already known to the ancient Greeks [3].

[6]These two problems cannot be reduced to neither linear nor quadratic equations.

[7]Squaring the circle is proved unsolvable by both straightedge and compass, and by origami.

[8]Note that trisecting an obtuse angle requires different construction steps.



(a) Specifying an angle to trisect, $\angle EAB$

(b) Applying **HA2** that brings A onto D

(c) Applying **HA2** that brings A onto G

(d) Applying **HA6** that brings G onto AE and A onto IH

Fig. 9. Construction steps for angle trisection using Abe's method.

S2 Axiom **HA2** is applied to points A and D, and B and C, respectively; in this way points A and B are brought onto points D and C, and the fold line that bisects segments AD i BC is constructed. Points of intersections of edges AD and BC with fold line are marked with G and F. It holds that G is a midpoint of the segment AD and F is a midpoint of the segment BC (see Figure 9(b)).

S3 Axiom **HA2** is applied to points A and G, and B and F, respectively; in this way points A and B are brought onto points G and F, and as a result the fold line that bisects segments AG and BF is constructed. Points of intersections of segments AG and BF with fold line are marked with I and H. It holds that I is a midpoint of the segment AG and H is a midpoint of the segment BF (see Figure 9(c)).

S4 Axiom **HA6** is applied to points G and A and lines AE and IH, and a fold line that brings G onto AE and A onto IH is constructed. This step creates points that are symmetric to the points G and A with respect to the fold line, which are marked with G_1 and A_1. Since folding preserves distances between points, by this folding the point I, which is a midpoint of the segment AG, is brought to the midpoint of the segment A_1G_1, marked with I_1. By constructing lines through points A and G_1, A and I_1, and A and A_1 angles $\angle EAI_1$, $\angle I_1AA_1$, and $\angle A_1AB$ are formed. It can be shown that each of them is equal to the third of the starting angle $\angle EAB$ (see Figure 9(d)), by proving that these three angles are mutually equal.

3.2. Proving construction correct

The proof of correctness of Abe's method for angle trisection can be performed manually or automatically. For instance, EOS (E-origami system) [7] automatically translates a sequence of origami construction steps and the goal to be proved into the set of equations and tries to solve them (using Gröbner basis method or cylindrical algebraic decomposition). Singular [4], as a computer algebra system for computations over polynomials, can also be used for proving construction correct. However, unlike EOS, in Singular a user has to do the translation of construction steps, obtained by folding, and the goal to be proved into the set of polynomials; then, a Gröbner basis method can be used. Here, for educational purposes, we will follow the second approach and discuss in detail the translation of each geometrical predicate into the corresponding set of polynomials.

In the text that follows, we will present two different approaches to prove construction correct: via tangent of the angle and via congruence of triangles.

Formulating construction steps using first-order logic formulae. Let us assume that the starting origami paper is square shaped and that the vertices of the square are marked with A, B, C, and D, respectively. Each two adjacent vertices of the square define the line, to which a corresponding side of the square belongs. The adjacent sides of a square are mutually perpendicular. Each step of the construction creates a set of constraints and they are as follows:

– The set of constraints corresponding to initial construction step **S1** can be written using previously introduced predicates in the following way:

$$\{onLine(A, AB),\ onLine(B, AB),\ onLine(C, CD),\ onLine(D, CD),\\ onLine(A, AD),\ onLine(D, AD),\ onLine(B, BC),\ onLine(C, BC),\\ perp(AB, BC),\ perp(AB, AD),\ perp(CD, BC),\ perp(CD, AD)\}$$

A point E is chosen as an arbitrary point of the side CD and an angle $\angle EAB$ is formed. The following formulae are added as constraints:

$$\{onLine(E, CD),\ onLine(A, AE),\ onLine(E, AE)\}$$

– In construction step **S2**, the bisector FG of segments AD and BC is constructed and the following formulae are added to the set of constraints:

$$\{onLine(G, AD),\ onLine(F, BC),\ onLine(G, GF),\ onLine(F, GF),\\ midpoint(G, A, D),\ midpoint(F, B, C),\ perp(GF, BC),\ perp(GF, AD)\}$$

– Similarly, in step **S3**, the bisector IH of segments AG and BF is constructed and the following constraints are added:

$$\{onLine(I, AG),\ onLine(H, BF),\ onLine(I, IH),\ onLine(H, IH),$$
$$midpoint(I, A, G),\ midpoint(H, B, F),\ perp(IH, BF),\ perp(IH, AG)\}$$

– Finally, after applying the construction step **S4**, the following formulae are added to the set of constraints:

$$\{symmetric(A, A_1, k), symmetric(G, G_1, k), onLine(A_1, IH), onLine(G_1, AE)\}$$

The union of previously described constraint sets will represent the common premises of the correctness conjecture for both approaches. It remains to formulate the rest of the premises, specific for a concrete way of proving correctness, and the conclusion of the conjecture, that the angle $\angle EAB$ is divided into three equal parts by lines AA_1 and AI_1.

As we already said, we prove the correctness of the construction in two different ways, by formulating two different correctness conjectures.

Approach via tangent of the angle. One way to prove that by the previous construction a trisection of the angle $\angle EAB$ is performed is by using trigonometry, more precisely using tangent function, following the ideas presented in [8].

Let us recall that the function $f(x) = \tan x$ is injective on $(0, \pi/2)$. Therefore, it follows that if for $x, y \in (0, \pi/2)$ holds $\tan x = \tan y$, then $x = y$ holds also. Since all three angles $\angle G_1AI_1, \angle I_1AA_1$, and $\angle A_1AB$ are acute, it suffices to show that the following equalities hold:

$$\tan(\angle G_1AI_1) = \tan(\angle I_1AA_1) = \tan(\angle A_1AB)$$

However, Singular does not support using trigonometric functions. Still, the value of the tangent function of the acute angle in right triangle can be expressed as a ratio of the length of its opposite cathetus and adjacent cathetus, which can be translated further into polynomial expressions over point coordinates. The only problem is that at this point the angles we are interested in do not belong to any right angle triangle (see Figure 9(d)).

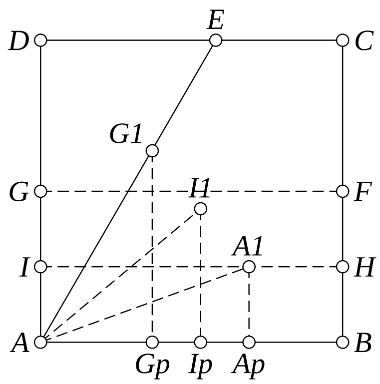


Fig. 10. Projecting points G_1, I_1, and A_1 onto segment AB.

In order to create corresponding right triangles, points G_1, I_1 and A_1 are projected onto the edge AB; that way points G_p, I_p and A_p are created (see Figure 10). Triangles $\triangle G_1AG_p$, $\triangle I_1AI_p$, and $\triangle A_1AA_p$ are right triangles with right angles at vertices G_p, I_p, and A_p, respectively.

Let us denote the angle $\angle A_1AA_p$ as α; the value $\tan\alpha$ can be calculated on the basis of the triangle $\triangle A_1AA_p$ using the following formula:

$$\tan\alpha = \frac{|A_1A_p|}{|AA_p|}. \tag{1}$$

However, the other two angles to which the conjecture applies are not the angles of the newly created right angle triangles. Nevertheless, it is possible to look at the problem from another perspective: instead of proving that the angles $\angle I_1AA_1$ and $\angle A_1AA_p$ are equal, we can prove that the angle $\angle I_1AI_p$ is twice the angle $\angle A_1AA_p$, i.e. that $\angle I_1AI_p = 2\alpha$. Similarly, it suffices to prove that $\angle G_1AG_p$ is three times the angle $\angle A_1AA_p$ or, equivalently, that $\angle G_1AG_p = 3\alpha$.

From $\triangle I_1AI_p$ and $\triangle G_1AG_p$ follow:

$$\tan(\angle I_1AI_p) = \frac{|I_1I_p|}{|AI_p|}, \quad \tan(\angle G_1AG_p) = \frac{|G_1G_p|}{|AG_p|}. \tag{2}$$

In order to express equations over trigonometry functions as polynomial equations, we use the following trigonometric identities:

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}, \qquad \tan(3\alpha) = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}. \tag{3}$$

Namely, we represent $\tan\alpha$ appearing in Equations (1) and (3) as a variable t and express constraints over it as a polynomial. By combining Equations (1), (2), and (3) we get:

$$\begin{aligned}
\frac{|A_1A_p|}{|AA_p|} &= \tan(\angle A_1AA_p) = \tan\alpha,\\
\frac{|I_1I_p|}{|AI_p|} &= \tan(\angle I_1AI_p) = \tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha},\\
\frac{|G_1G_p|}{|AG_p|} &= \tan(\angle G_1AG_p) = \tan(3\alpha) = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}.
\end{aligned}$$

Simplifying the previous equations and using variable t instead of $\tan\alpha$ we get the following polynomial equations:

$$\begin{aligned}
|AA_p|t - |A_1A_p| &= 0,\\
|I_1I_p|t^2 + 2|AI_p|t - |I_1I_p| &= 0,\\
|AG_p|t^3 - 3\cdot|G_1G_p|t^2 - 3|AG_p|t + |G_1G_p| &= 0.
\end{aligned} \tag{4}$$

The first of equations in (4) defines a variable t and therefore is included into the set of premises of the correctness conjecture, while the last two equations are statements we tend to prove and therefore correspond to the conclusions of correctness conjecture. Notice that the last equation involves third degree polynomial in t and, as we already said, such equation is not solvable using straightedge and compass, but is solvable using origami.

Approach via congruence of triangles. Another way of proving the correctness of considered construction is by proving the congruence of corresponding triangles.

Two triangles are congruent if and only if their corresponding sides are equal (lengthwise) and their corresponding angles are equal. Usually, it suffices to show that three corresponding elements of two triangles are equal to deduce their congruence, such as the three sides, the three angles or a combination of sides and angles. For the purpose of this paper we will use the following two congruence postulates:

- Side–Angle–Side (**SAS**) which states that if two pairs of sides and included angle are equal then the triangles are congruent,
- Side–Side–Angle (**SSA**) which states that if two pairs of sides and non–included angle are equal then the triangles are congruent.

Let us first denote point A_p the orthogonal projection of the point A_1 onto the segment AB. We will try to deduce that $\triangle G_1AI_1 \cong \triangle AA_1I_1 \cong \triangle A_1AA_p$ from where it would follow that the corresponding angles $\angle G_1AI_1, \angle I_1AA_1, \angle A_1AA_p$ are equal to each other and therefore equal to the third of $\angle EAB$ (see Figure 11).

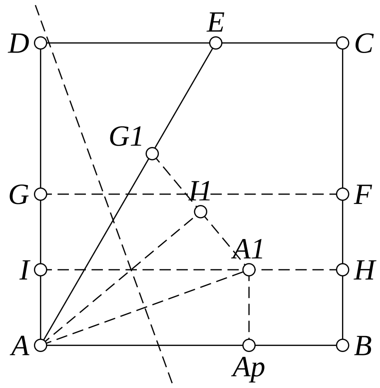


Fig. 11. Right angle triangles $\triangle G_1AI_1$, $\triangle AA_1I_1$, $\triangle A_1AA_p$ defined by constructed points.

However, proving the congruence of triangles in Singular is not straightforward. Namely, Singular does not support the congruence functions, and therefore we have to define the conjecture as equality of certain triangle elements. By doing so, we can conclude that the certain triangles are congruent and, therefore, that their corresponding sides and angles are equal.

Let us first consider the triangles $\triangle G_1AI_1$ and $\triangle AA_1I_1$. To conclude that these two triangles are congruent we will use the **SAS** congruence postulate. Since these two triangles have the common side AI_1 it is sufficient to prove the following:

$$AI_1 \perp G_1A_1, \quad |G_1I_1| = |I_1A_1|. \tag{5}$$

Line A_1I is a perpendicular bisector of the segment AG. Since the distances are preserved by folding, as well as the angles, it follows that the line AI_1 is a perpendicular bisector of the segment A_1G_1 and, therefore, Equation (5) should hold.

Similarly, to deduce that $\triangle AA_1I_1$ and $\triangle A_1AA_p$ are congruent we will use the **SSA** congruence postulate. Since these two triangles share the side AA_1 it is sufficient to prove the following:

$$|I_1A_1| = |A_1A_p|, \quad A_1A_p \perp AA_p. \tag{6}$$

As we have already said, folding preserves distances between points. Therefore points A_1 and I_1 are at the same distance as points A and I. Moreover, points A_1 and A_p are the at the same distance as A and I since A_1 is on the line IH and A_p is is the orthogonal projection of A_1 onto BC.

From the congruence of these two pairs of triangles follows that their corresponding angles are equal, that is:

$$\angle G_1AI_1 = \angle I_1AA_1 = \angle A_1AA_p = \angle EAB/3.$$

The conditions given by Equations (5) and (6) are easily formulated as a corresponding formulae over point coordinates in Singular.

4. Implementation

So far we have described the origami construction as a sequence of first-order formulae specifying the premises and the conclusion of the conjecture being proved. In order to use Gröbner basis method to prove the conjecture, these formulae are translated into the set of algebraic equations following the approach described in Section 2.1.

First we need to place the elements of the starting configuration into the coordinate plane. Notice that the properties of geometric figures such as lines and angles do not change under the effect of translation and rotation in the Euclidean plane. So when introducing Cartesian coordinates, we can assign the given points some suitable coordinates, which simplify the following calculations. We can place the lower-left corner of the paper into the origin, and assume that the size of the paper is equal to a. In such setting, the corner-points of the paper have coordinates $A = (0, 0)$, $B = (a, 0)$, $C = (a, a)$ and $D = (0, a)$, where $a \in \mathbb{R}$. We are only considering non-degenerated case where $a \neq 0$[9].

Proving correctness using Gröbner basis method requires a great deal of symbolic manipulations and for this purpose we use the computer algebra tool Singular. All calculations over polynomials are done in the ring of real numbers over degree

[9] The condition $a \neq 0$ can be expressed as polynomial using formula: $\mathtt{M}(\neg(a = 0)) = \{a\epsilon - 1 = 0\}$, where ϵ is a new variable.

lexicographical ordered variables. Variables over which the polynomials are formulated correspond to the coordinates of points, coefficients of lines, and tangent of the trisected angle.

After specifying a variable ring, every constraint from the set of premises, formulated as a polynomial, is declared as a variable of a type `poly`. An ideal Idl over these polynomials is generated and a Gröbner basis GRB of ideal Idl is calculated. For this purpose function `groebner` is used. Then each of the polynomials corresponding to the conclusion of the conjecture is reduced with respect to the Gröbner basis GRB using function `reduce`. If the return value is 0, then the conclusion belongs to the ideal Idl, thus proving the correctness conjecture.

The predicates introduced in Section 2.1 are translated into the set of Singular procedures and they are collected within a Singular library `origami.lib` (see Figure 12), which is then included in the main program (see Figure 13).

```
// returns a vector of point coordinates
proc makePoint(list x){
    return([x(1), x(2)]);
}

// returns a vector of line coefficients
proc onLine(list l){
    return([l(1), l(2), l(3)]);
}

// checks if a point P lies on a line l
proc onLine(vector P, vector l){
    poly f = l[1]*P[1] + l[2]*P[2] + l[3];
    return(f);
}
```

Fig. 12. Fragment of the library `origami.lib`.

```
LIB "origami.lib";
ring r = real, (ab(1..3), a(1..2), b(1..2)), Dp;

vector A = makePoint(a(1..2));
vector B = makePoint(b(1..2));
vector AB = makeLine(ab(1..3));

poly f1 = onLine(A, AB); // AB contains A
poly f2 = onLine(B, AB); // AB contains B
```

Fig. 13. Using Singular functions defined in library `origami.lib`.

4.1. Proof via the tangent of the angle

The first approach to proving correctness conjecture, via the trigonometry transformations, requires proving the last two statements given by Equation (4).

Points G_1, I_1 and A_1 were projected on edge AB, therefore projection points G_p, I_p and A_p have the following coordinates $G_p = (x_{G_1}, 0)$, $I_p = (x_{I_1}, 0)$ and $A_p = (x_{A_1}, 0)$. Line segments appearing in these equations are either horizontal or vertical, therefore their lengths are: $|AA_p| = x_{A_1}$, $|A_1A_p| = y_{A_1}$, $|AI_p| = x_{I_1}$, $|I_1I_p| = y_{I_1}$, $|AG_p| = x_{G_1}$, $|G_1G_p| = y_{G_1}$. The tangent of an angle $\angle A_1AA_p$ is denoted by variable `t` and defined as a ratio of the length of its opposite and adjacent catheti in triangle $\triangle A_1AA_p$.

```
1 ring r = real, (a1(1..2), i1(1..2), g1(1..2), t), Dp;
2
3 poly f = a1[1]*t - a1[2];
4 poly g1 = i1[2]*t^2 + 2*i1[1]*t - i1[2];
5 poly g2 = g1[1]*t^3 - 3*g1[2]*t^2 - 3*g1[1]*t + g1[2];
```

Fig. 14. Formulation of constraints over tangents in Singular.

In the third line of the Figure 14 a tangent of angle $\angle A_1AA_p$ of the triangle $\triangle A_1AA_p$ is defined. Polynomial f will be added to the set of premises of the correctness conjecture. In the fourth and fifth line the conclusions of the conjecture to be proved are defined.

4.2. Proof via the congruence of triangles

The second approach to proving correctness conjecture, via the congruence of triangles, requires proving statements given by Equations (5) and (6). Recall that these statements are obtained by applying postulate **SAS** on triangles $\triangle G_1AI_1$ and $\triangle AA_1I_1$ and postulate **SSA** on triangles $\triangle AA_1I_1$ and $\triangle A_1AA_p$.

```
LIB "origami.lib";
ring r = real,(dg1i1, di1a1, dai1, g1a1(1..3), ai1(1..3)), Dp;

proc SAS(def da1, def da2, vector m, vector n,
         def db1, def db2, ideal GRB){
    poly g1 = reduce(da1 - da2, GRB);
    poly g2 = reduce(perp(m, n), GRB);
    poly g3 = reduce(db1 - db2, GRB);
    return(g1 == 0 && g2 == 0 && g3 == 0);
}
vector G1A1 = makeLine(g1a1(1..3));
vector AI1 = makeLine(ai1(1..3));
// Groebner basis (GRB) is already calculated
SAS(dg1i1, di1a1, AI1, G1I1, dai1, dai1, GRB);
```

Fig. 15. Formulation of constraints corresponding to triangle congruence in Singular.

In Figure 15, an application of **SAS** postulate to triangles $\triangle G_1AI_1$ and $\triangle AA_1I_1$ is presented. Note that in this case side A_1A as the joint side of these triangles is trivially equal to itself. Function for checking if the two triangles are congruent according to **SSA** postulate is implemented in a similar way.

Developed library and complete proofs of correctness in Singular can be found on: `https://github.com/Dara123M/origamiSingular`.

5. Conclusion and further work

In this paper we have illustrated the way origami constructions are performed, with the example of angle trisection problem. Additionally, the correctness of performed construction is proved using computer algebra tool Singular.

The angle trisection is chosen as an example problem as it is one of the famous construction problems, but also easy to be solved using origami. Therefore we find it a good choice for students to start learning constructions by origami. Other construction problems can be solved and proved correct using this method in a similar way.

We tried to approach this problem from educational perspective, but also to keep high level of mathematical rigor. Namely, the correctness of solution to construction problem is not guaranteed by construction and there is a need to independently prove its correctness.

The simplicity of origami constructions is what gives it its charm. We believe that origami is worth considering as an alternative learning method, not only for teaching math, but also to improve other skills, such as creativity, deduction, thoroughness, etc. We hope that this paper will inspire other math teacher to try origami constructions in class and make their classes a bit different than the one we are used to.

ACKNOWLEDGMENT. This research is supported by the Serbian Ministry of Education, Science and Technological Development through the University of Belgrade, Faculty of Mathematics, grant number: 451-03-47/2023-01/ 200104.

The authors are thankful to the anonymous referee for helpful suggestions that helped us to improve the paper.

REFERENCES

[1] B. Buchberger, T. Ida, *Origami Theorem Proving*, SFB Scientific Computing Technical Report, 2003.

[2] B. Buchberger, M. Kauers, *Buchberger's algorithm*, Scholarpedia **6**,(10):7764, 2011, revision #126537.

[3] R. Courant, H. Robbins, *What Is Mathematics?: An Elementary Approach to Ideas and Methods*, 2nd ed., Oxford University Press, 1996.

[4] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 4-3-0 – *A computer algebra system for polynomial computations*, 2022, `http://www.singular.uni-kl.de`.

[5] K. O. Geddes, S. R. Czapor, G. Labahn, *Algorithms for Computer Algebra*, Kluwer Academic Publishers, 1992.

[6] F. Ghourabi, T. Ida, H. Takahashi, A. Kasem, *Logical and algebraic view of Huzita's origami axioms with applications to computational origami*, Proc. ACM Symposium on Applied Computing, March 2007, pp. 767–772.

[7] T. Ida, H. Takahashi, M. Marin, A. Kasem, F. Ghourabi, *Computational Origami System Eos*, Proc. 4th Int. Conf. Origami, Sci., Math. and Education, August 2009,

[8] T. Ida, D. Tepeneu, B. Buchberger, J. Robu, *Proving and Constraint Solving in Computational Origami*, In: B. Buchberger, J. Campbell (eds.), *Artificial Intelligence and Symbolic Computation*, AISC 2004, Lecture Notes in Computer Science, vol. 3249, Springer, Berlin, Heidelberg, pp. 132–142.

[9] R. J. Lang, *Origami and geometric constructions*, Tokyo: Gallery origami house, **68**, 2010.

[10] V. Marinković, P. Janičić, *Towards understanding triangle construction problems*, In: *Intelligent Computer Mathematics - CICM 2012*, Lecture Notes in Computer Science **7362**, 2012, pp. 126–141.

[11] V. Marinković, *ArgoTriCS – Automated triangle construction solver*, J. Experimental & Theoretical Artificial Intelligence, **29**(2) (2017), 247–271.

[12] G. E. Martin, *Geometric Constructions*, Springer Science & Business Medi, 2012.

[13] L. Redman, *Algebraic methods for proving geometric theorems*, Electronic Theses, Projects, and Dissertations, 923, 2019.

[14] K. F. Rivas, *Geometric theorem proving using the Groebner basis algorithm*, Theses Digitization Project, 3531, 2009.

[15] P. Schreck, *On the mechanization of straightedge and compass constructions*, J. Systems Science and Complexity **32**, 2 (2019), 124–149.

[16] F. von Lindemann, *Über die Zahl π*, Math. Annalen **20** (1882), 213–225.

[17] P. L. Wantzel, *Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas*, J. de Mathématiques Pures et Appliquées, 1e série, **2** (1837), 366–372.

[18] Wen-Tsun Wu, *Basic principles of mechanical theorem proving in elementary geometries*, J. Automated Reasoning, **2** (1984), 221–252.

D.M.: Faculty of Mathematics, University of Belgrade, Serbia

ORCID: 0009-0006-6456-9329

E-mail: `dara.milojkovic@matf.bg.ac.rs`

V.M.: Faculty of Mathematics, University of Belgrade, Serbia

ORCID: 0000-0003-0526-899X

E-mail: `vesna.marinkovic@matf.bg.ac.rs`

Received: 19.07.2024, in revised form 12.09.2024.

Accepted: 15.09.2024