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THE LIMIT OF THE INCREMENTS OF THE HÖLDER MEANS OF ASYMPTOTICALLY ARITHMETIC SEQUENCES

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Abstract. We call a sequence of real numbers, $\{a_n\}_{n\geq 1}$, an asymptotically arithmetic sequence, if its increment $a_{n+1}-a_n$ approaches a real number d, as $n \to \infty$. For each $p \in [-\infty, \infty]$, we compute the limit of the increment $H_p(a_1, \dots, a_n, a_{n+1}) - H_p(a_1, \dots, a_n)$, of the p-Hölder mean sequence, $\{H_p(a_1, \dots, a_n)\}_{n\geq 1}$, of an asymptotically arithmetic sequence $\{a_n\}_{n\geq 1}$, with positive terms. Moreover, for $p \leq -1$, we not only show that this limit is 0, but we also compute the rate with which the increment approaches zero.

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1. Introduction

It is known that given two sequences of reals numbers $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, such that $0 < b_1 < b_2 < \cdots$ and $b_n \to \infty$, as $n \to \infty$, we have

$$\liminf_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \leq \liminf_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}. \tag{1.1}$$

As a corollary of this inequality, we obtain the following Stolz-Cesàro theorem.

THEOREM 1.1. *Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers, and $\{b_n\}_{n\geq 1}$ an increasing sequence of real numbers tending to ∞, as $n \to \infty$. If the limit of the ratio of their increments*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l,$$

exists in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, then the limit $\lim_{n\to\infty} a_n/b_n$ also exists in $\overline{\mathbb{R}}$, and we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

If one takes, for all $n \geq 1$, $a_n := \ln(A_n)$ and $b_n := n$, where $\{A_n\}_{n\geq 1}$ is an arbitrary sequence of positive real numbers, then inequality (1.1) is equivalent to

$$\liminf_{n\to\infty} \frac{A_{n+1}}{A_n} \leq \liminf_{n\to\infty} \sqrt[n]{A_n} \leq \limsup_{n\to\infty} \sqrt[n]{A_n} \leq \limsup_{n\to\infty} \frac{A_{n+1}}{A_n}. \tag{1.2}$$

As a corollary of this inequality, we obtain the following D'Alembert theorem.

THEOREM 1.2. *Let* $\{A_n\}_{n\geq 1}$ *be an arbitrary sequence of positive real numbers. If the limit of the ratio of the consecutive terms*

$$\lim_{n\to\infty}\frac{A_{n+1}}{A_n}=l,$$

exists in $[0,\infty]$*, then the limit* $\lim_{n\to\infty}\sqrt[n]{A_n}$ *also exists in* $[0,\infty]$*, and we have*

$$\lim_{n\to\infty}\sqrt[n]{A_n}=l.$$

A classical application of D'Alembert theorem is the following limit

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}. \tag{1.3}$$

The reciprocals of the Stolz-Cesàro theorem and D'Alembert theorem do not hold in general.

Starting from the well-known limit (1.3), we can try to go backwards, and assuming that the reciprocal of Stolz-Cesàro theorem holds for $a_n:=\sqrt[n]{n!}$ and $b_n:=n$, for all $n\geq 1$, we may ask whether the sequence $\{(a_{n+1}-a_n)/(b_{n+1}-b_n)\}_{n\geq 1}$ converges to $1/e$. This sequence is

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!},$$

for all $n\geq 1$. T. Lalescu proved that, in this particular case, the reciprocal of Stolz-Cesàro theorem holds and we have

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{1}{e}.$$

Since then, the sequence

$$L_n:=\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!},$$

for all $n\geq 1$, has been called the *Lalescu sequence*.

Starting from the observation that, for all $n\geq 1$,

$$\sqrt[n]{n!}=\sqrt[n]{1\cdot 2\cdots n}=\sqrt[n]{a_1\cdot a_2\cdots a_n},$$

where $\{a_k\}_{k\geq 1}:=\{k\}_{k\geq 1}$ forms an arithmetic sequence with the common difference $a_{k+1}-a_k=k+1-k=1$, we generalize this problem in two ways.

First, we consider an "asymptotically arithmetic sequence of positive limit difference d", that means a sequence with (strictly) positive terms, $\{a_n\}_{n\geq 1}$, such that there exists

$$\lim_{n\to\infty}(a_{n+1}-a_n)=d>0,$$

and compute the limit

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_1\cdots a_n\cdot a_{n+1}}-\sqrt[n]{a_1\cdot a_2\cdots a_n}\right). \tag{1.4}$$

Second, observing that, for all $n \geq 1$, $\sqrt[n]{a_1 \cdot a_2 \cdots a_n}$ is the geometric mean of the positive numbers a_1, a_2, $\ldots$, a_n, and the geometric mean is the 0-Hölder mean $H_0(a_1, a_2, \ldots, a_n)$ of the same numbers, for all $p \in [-\infty, \infty]$, we compute

$$\lim_{n\to\infty} \left[H_p\left(a_1, \ldots, a_n, a_{n+1}\right) - H_p\left(a_1, a_2, \ldots, a_n\right)\right], \tag{1.5}$$

where $H_p(a_1, a_2, \ldots, a_n)$ denotes the p-Hölder mean of the positive numbers a_1, a_2, $\ldots$, a_n, for all $n \geq 1$.

We are not the first people to consider this problem. The limit (1.5) was already computed, when $p \in [0, \infty]$, in [1]. Our contribution consists in computing this limit for $p \in [-\infty, 0)$, and in the case when $p \in (-\infty, -1]$, not only we will see that this limit is 0, but we will also see how fast the increments of the Hölder means are converging to 0.

The paper is structured as follows. In Section 2, we compute the limit (1.4), and in Section 3, we calculate the limit (1.5), for all $p \in [-\infty, \infty] \setminus \{0\}$, for an asymptotically arithmetic sequence of positive limit difference d.

2. The limit of the increment of the geometric mean sequence of an asymptotically arithmetic sequence with positive terms

We have the following theorem.

THEOREM 2.1. *Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers such that the sequence $\{a_{n+1} - a_n\}_{n\geq 1}$ is convergent and $d := \lim_{n\to\infty}(a_{n+1} - a_n) > 0$. Then we have*

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_1 a_2 \cdots a_{n+1}} - \sqrt[n]{a_1 a_2 \cdots a_n} \right) = \frac{d}{e}.$$

Proof. Since $\lim_{n\to\infty}(a_{n+1} - a_n) = d$, it follows from Stolz-Cesàro theorem that

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{(n+1) - n} = d.$$

Since $d > 0$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{n} \cdot \lim_{n\to\infty} n = d \cdot \infty = \infty.$$

Using D'Alembert theorem we have

$$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt[n]{a_1 a_2 \cdots a_n}}{a_n} &= \lim_{n\to\infty} \sqrt[n]{\frac{a_1 a_2 \cdots a_n}{a_n^n}} \\ &= \lim_{n\to\infty} \left[\left(\frac{a_1 a_2 \cdots a_n a_{n+1}}{a_{n+1}^{n+1}} \right) / \left(\frac{a_1 a_2 \cdots a_n}{a_n^n} \right) \right] \\ &= \lim_{n\to\infty} \left(\frac{a_n}{a_{n+1}} \right)^n = 1 / \left[\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n} \right)^n \right]. \end{aligned} \tag{2.1}$$

Let us observe that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{a_{n+1} - a_n}{a_n}\right) = 1 + \frac{d}{\infty} = 1.$$

The limit from the denominator of the fraction from equation (2.1) is a 1^∞ limit. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt[n]{a_1 a_2 \cdots a_n}}{a_n} &= \frac{1}{\lim\limits_{n\to\infty} \left[\left(1 + \frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right]^{\frac{n(a_{n+1}-a_n)}{a_n}}} \\
&= \frac{1}{\left[\lim\limits_{n\to\infty} \left(1 + \frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right]^{\left(\lim\limits_{n\to\infty} \frac{n}{a_n}\right) \lim\limits_{n\to\infty} (a_{n+1}-a_n)}} \\
&= \frac{1}{e^{\frac{1}{d}\cdot d}} = \frac{1}{e},
\end{aligned}$$

since $(a_{n+1} - a_n)/a_n \to d/\infty = 0$, as $n \to \infty$, and $\lim_{x\to 0}(1 + x)^{1/x} = e$. We have

$$\begin{aligned}
&\lim_{n\to\infty} \left(\sqrt[n+1]{a_1 a_2 \cdots a_{n+1}} - \sqrt[n]{a_1 a_2 \cdots a_n}\right) \\
&\qquad = \lim_{n\to\infty} \sqrt[n+1]{a_1 a_2 \cdots a_{n+1}} \left[1 - \frac{(a_1 a_2 \cdots a_n)^{\frac{1}{n} - \frac{1}{n+1}}}{a_{n+1}^{\frac{1}{n+1}}}\right].
\end{aligned}$$

Thus, since $(1/n) - 1/(n + 1) = 1/[n(n + 1)]$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left(\sqrt[n+1]{a_1 a_2 \cdots a_{n+1}} - \sqrt[n]{a_1 a_2 \cdots a_n}\right) \\
&\quad = \left[\lim_{n\to\infty} \frac{\sqrt[n+1]{a_1 a_2 \cdots a_{n+1}}}{a_{n+1}}\right] \cdot \left[\lim_{n\to\infty} \frac{a_{n+1}}{n+1}\right] \\
&\qquad \times \lim_{n\to\infty} \left\{ (n+1) \left[1 - \left(\frac{\sqrt[n]{a_1 a_2 \cdots a_n}}{a_{n+1}}\right)^{\frac{1}{n+1}}\right]\right\} \\
&\quad = \frac{1}{e} \cdot d \cdot \lim_{n\to\infty} \left[\frac{1 - \left(\frac{\sqrt[n]{a_1 a_2 \cdots a_n}}{a_{n+1}}\right)^{\frac{1}{n+1}}}{\frac{1}{n+1}}\right] = \frac{d}{e} \cdot \lim_{n\to\infty} \frac{1 - y_n^{\frac{1}{n+1}}}{\frac{1}{n+1}},
\end{aligned}$$

where

$$\begin{aligned}
y_n := \frac{\sqrt[n]{a_1 a_2 \cdots a_n}}{a_{n+1}} &= \frac{\sqrt[n]{a_1 a_2 \cdots a_n}}{a_n} \cdot \frac{a_n}{n} \cdot \frac{n+1}{a_{n+1}} \cdot \frac{n}{n+1} \\
&\to \frac{1}{e} \cdot d \cdot \frac{1}{d} \cdot 1 = \frac{1}{e}.
\end{aligned}$$

Let $x_n := \ln(y_n)/(n + 1) \to 0$, as $n \to \infty$. We have

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_1 a_2 \cdots a_{n+1}} - \sqrt[n]{a_1 a_2 \cdots a_n}\right) = \frac{d}{e} \cdot \lim_{n\to\infty} \frac{1 - y_n^{\frac{1}{n+1}}}{\frac{1}{n+1}}$$

$$
\begin{aligned}
&= \frac{d}{e} \cdot \lim_{n\to\infty} \frac{1 - e^{\frac{\ln(y_n)}{n+1}}}{\frac{\ln(y_n)}{n+1}} \cdot \lim_{n\to\infty} \ln(y_n) = \frac{d}{e} \cdot \lim_{n\to\infty} \frac{1 - e^{x_n}}{x_n} \cdot \ln\left(\frac{1}{e}\right) \\
&= \frac{d}{e} \cdot \lim_{n\to\infty} \frac{1 - e^{x_n}}{x_n} \cdot (-1) = \frac{d}{e} \cdot \lim_{n\to\infty} \frac{e^{x_n} - 1}{x_n} \\
&= \frac{d}{e} \cdot \lim_{n\to\infty} \frac{e^{x_n} - e^0}{x_n - 0} = \frac{d}{e} \cdot f'(0),
\end{aligned}
$$

where $f(x) := e^x$. Since $f'(0) = 1$, we conclude that

$$
\lim_{n\to\infty} \left(\sqrt[n+1]{a_1 a_2 \cdots a_{n+1}} - \sqrt[n]{a_1 a_2 \cdots a_n} \right) = \frac{d}{e}. \quad \blacksquare
$$

3. The limit of the increment of the Hölder mean sequence of an asymptotically arithmetic sequence with positive terms

In this section we show that the sequence of the p-Hölder means, of an asymptotically arithmetic sequence with positive terms of positive limit difference d, converges to $d[(1+p)^+]^{-1/p}$, for all $p \in [-\infty, \infty] \setminus \{0\}$, where $x^+ := \max\{x,\, 0\}$, for any real number x. For any positive numbers $x_1,\, x_2,\, \ldots,\, x_n$, and for any number $p \in [-\infty, \infty]$, we define

$$
H_p(x_1, x_2, \ldots, x_n) = \begin{cases}
\left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{1/p}, & \text{if } p \in \mathbb{R} \setminus \{0\}, \\
\lim_{p\to 0} \left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{1/p} = \sqrt[n]{x_1 x_2 \cdots x_n}, & \text{if } p = 0, \\
\lim_{p\to \infty} \left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{1/p} = \max\{x_1, x_2, \ldots, x_n\}, & \text{if } p = \infty, \\
\lim_{p\to -\infty} \left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{1/p} = \min\{x_1, x_2, \ldots, x_n\}, & \text{if } p = -\infty,
\end{cases}
$$

and call $H_p(x_1, x_2, \ldots, x_n)$ the *p-Hölder mean* of $x_1,\, x_2,\, \ldots,\, x_n$.

We have the following theorem.

THEOREM 3.1. *Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers such that the sequence $\{a_{n+1} - a_n\}_{n\geq 1}$ is convergent and $d := \lim_{n\to\infty}(a_{n+1} - a_n) > 0$. Then, we have*

$$
\lim_{n\to\infty} [H_p(a_1, \ldots, a_n, a_{n+1}) - H_p(a_1, a_2, \ldots, a_n)] = \begin{cases}
\frac{d}{(1+p)^{1/p}}, & \text{if } p \in (-1, \infty) \setminus \{0\}, \\
\frac{d}{e}, & \text{if } p = 0, \\
d, & \text{if } p = \infty, \\
0, & \text{if } p \in [-\infty, -1].
\end{cases} \tag{3.1}
$$

Moreover, we have

$$\lim_{n\to\infty}\{\ln(n)\left[H_{-1}\left(a_1,\ldots,a_n,a_{n+1}\right)-H_{-1}\left(a_1,a_2,\ldots,a_n\right)\right]\}=d,$$

and for all $-\infty<p<-1$*, we have*

$$\lim_{n\to\infty}\{n^{1+(1/p)}\left[H_p\left(a_1,\ldots,a_n,a_{n+1}\right)-H_p\left(a_1,a_2,\ldots,a_n\right)\right]\}=\frac{1}{q\|\{1/a_n\}\|_q},$$

where $q:=-p$ *and* $\|\{1/a_n\}\|_q$ *is the* l^q*-norm of the sequence* $\{1/a_n\}_{n\geq 1}$*, defined as*

$$\|\{1/a_n\}\|_q:=\left[\sum_{n=1}^{\infty}\frac{1}{a_n^q}\right]^{1/q}.$$

Proof. We have already proven formula (3.1) for $p=0$ in the previous section, and so we may assume that $p\neq 0$. We distinguish between five cases:

Case 1. If $p=\infty$, then since $\lim_{n\to\infty}(a_{n+1}-a_n)=d>0$, there exists $N_1\in\mathbb{N}$, such that for all $n\geq N_1$, we have $a_{n+1}-a_n>0$, which means $a_{N_1}<a_{N_1+1}<a_{N_1+2}<\cdots$. Since $\lim_{n\to\infty}a_n=\infty$, there exists $N_2\in\mathbb{N}$, $N_2\geq N_1$, such that, for all $n\geq N_2$, we have

$$a_n>\max\{a_1,a_2,\ldots,a_{N_1}\}.$$

Therefore, for all $n\geq N_2$, we have

$$H_\infty\left(a_1,a_2,\ldots,a_n\right)=\max\{a_1,a_2,\ldots,a_n\}=a_n.$$

Thus, we have

$$\lim_{n\to\infty}\left[H_\infty\left(a_1,\ldots,a_n,a_{n+1}\right)-H_\infty\left(a_1,a_2,\ldots,a_n\right)\right]=\lim_{n\to\infty}\left(a_{n+1}-a_n\right)=d.$$

Case 2. If $p\in(-1,\infty)\setminus\{0\}$, then the sequence $\{n^{p+1}\}_{n\geq 1}$ increases and tends to $+\infty$, as $n\to\infty$. This allows us to apply Stolz-Cesàro theorem and obtain

(3.2)
$$\begin{aligned}\lim_{n\to\infty}\frac{H_p^p(a_1,a_2,\ldots,a_p)}{n^p}&=\lim_{n\to\infty}\frac{a_1^p+a_2^p+\cdots+a_n^p}{n^{p+1}}\\&=\lim_{n\to\infty}\frac{(a_1^p+\cdots+a_n^p+a_{n+1}^p)-(a_1^p+a_2^p+\cdots+a_n^p)}{(n+1)^{p+1}-n^{p+1}}\\&=\lim_{n\to\infty}\frac{a_{n+1}^p}{(n+1)^p}\cdot\lim_{n\to\infty}\frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}.\end{aligned}$$

For each $n\geq 1$, applying Lagrange Mean Value theorem to the function $f(x)=x^{p+1}$ on the interval $[n,n+1]$, we conclude that there exists a number $c_n\in(n,n+1)$, such that

$$\frac{(n+1)^{p+1}-n^{p+1}}{n+1-n}=(p+1)c_n^p.$$

Thus formula (3.2) becomes

$$\begin{aligned}(3.3)\quad \lim_{n\to\infty}\frac{H_p^p(a_1,a_2,\ldots,a_p)}{n^p} &= \lim_{n\to\infty}\frac{a_{n+1}^p}{(n+1)^p}\cdot\lim_{n\to\infty}\frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}\\ &= \left[\lim_{n\to\infty}\frac{a_{n+1}}{n+1}\right]^p\cdot\lim_{n\to\infty}\frac{(n+1)^p}{(p+1)c_n^p} = d^p\cdot\frac{1}{p+1}\left[\lim_{n\to\infty}\frac{n+1}{c_n}\right]^p\\ &= d^p\cdot\frac{1}{p+1}\cdot 1^p = \frac{d^p}{p+1},\end{aligned}$$

since we saw in the previous section that $\lim_{n\to\infty}(a_n/n) = d$, and for all $n \geq 1$ we have

$$\frac{n+1}{n+1} < \frac{n+1}{c_n} < \frac{n+1}{n},$$

and so, by the Squeeze theorem, it follows that $\lim_{n\to\infty}[(n+1)/c_n] = 1$.

Using the same number N_2 as in the previous case, which has the property that for all $n \geq N_2$, we have $a_{n+1} > \max\{a_1, a_2, \ldots, a_n\}$, we conclude that, for all $n \geq N_2$:

- if $0 < p < \infty$, then

$$\begin{aligned}&\frac{a_1^p+\cdots+a_n^p+a_{n+1}^p}{n+1} > \frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\\ \iff\ & H_p^p\left(a_1,\ldots,a_n,a_{n+1}\right) > H_p^p\left(a_1,a_2,\ldots,a_n\right).\end{aligned}$$

- if $-1 < p < 0$, then

$$\begin{aligned}&\frac{a_1^p+\cdots+a_n^p+a_{n+1}^p}{n+1} < \frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\\ \iff\ & H_p^p\left(a_1,\ldots,a_n,a_{n+1}\right) < H_p^p\left(a_1,a_2,\ldots,a_n\right).\end{aligned}$$

To shorten the notation, we will define, for all $n \geq 1$,

$$H_{p,n} := H_p\left(a_1, a_2, \ldots, a_n\right).$$

Thus, for all $n \geq N_2$, applying the Lagrange Mean Value theorem to the function $g(x) = x^{1/p}$ on the interval $[H_{p,n}^p, H_{p,n+1}^p]$, if $p > 0$, or on the interval $[H_{p,n+1}^p, H_{p,n}^p]$, if $p < 0$, we conclude that there exists a number ξ_n strictly in between $H_{p,n}^p$ and $H_{p,n+1}^p$, such that

$$\begin{aligned}&H_{p,n+1}-H_{p,n} = \frac{1}{p}\xi_n^{(1-p)/p}\left[H_{p,n+1}^p - H_{p,n}^p\right]\\ \iff\ & H_{p,n+1}-H_{p,n} = \frac{1}{p}\cdot\left[\left(\frac{\xi_n}{n^p}\right)^{(1-p)/p}\cdot\frac{H_{p,n+1}^p-H_{p,n}^p}{n^{p-1}}\right].\end{aligned}$$

Therefore, we have

(3.4)

$$
\begin{aligned}
&\lim_{n\to\infty}\left[H_{p,n+1}-H_{p,n}\right]\\
&= \frac{1}{p}\cdot\lim_{n\to\infty}\left[\left(\frac{\xi_n}{n^p}\right)\right]^{(1-p)/p}\cdot\lim_{n\to\infty}\frac{\frac{a_1^p+\cdots+a_n^p+a_{n+1}^p}{n+1}-\frac{a_1^p+a_2^p+\cdots+a_n^p}{n}}{n^{p-1}}\\
&= \frac{1}{p}\cdot\lim_{n\to\infty}\left[\left(\frac{\xi_n}{n^p}\right)\right]^{(1-p)/p}\cdot\lim_{n\to\infty}\frac{na_{n+1}^p-(a_1^p+\cdots+a_n^p)}{n^p(n+1)}\\
&= \frac{1}{p}\cdot\lim_{n\to\infty}\left[\left(\frac{\xi_n}{n^p}\right)\right]^{(1-p)/p}\cdot\lim_{n\to\infty}\frac{na_{n+1}^p-(a_1^p+\cdots+a_n^p)}{n^{p+1}}\cdot\lim_{n\to\infty}\frac{n}{n+1}\\
&= \frac{1}{p}\cdot\left[\frac{d^p}{p+1}\right]^{(1-p)/p}\cdot\lim_{n\to\infty}\frac{na_{n+1}^p-(a_1^p+\cdots+a_n^p)}{n^{p+1}}\cdot 1\\
&= \frac{d^{1-p}}{p(1+p)^{(1-p)/p}}\cdot\lim_{n\to\infty}\frac{na_{n+1}^p-(a_1^p+\cdots+a_n^p)}{n^{p+1}}.
\end{aligned}
$$

since $\lim_{n\to\infty}(\xi_n/n^p)=d^p/(p+1)$ as it will be explained below.

The fraction ξ_n/n^p is in between the numbers $H_{p,n}^p/n^p$ and $H_{p,n+1}^p/n^p = H_{p,n+1}^p/(n+1)^p\cdot[(n+1)/n]^p$, and since we know, from formula (3.3), that $H_{p,n}^p/n^p\to d^p/(p+1)$, as $n\to\infty$, it follows from the Squeeze theorem that $\xi_n/n^p\to d^p/(p+1)$, as $n\to\infty$.

Since $p+1>0$, the sequence $\{n^{p+1}\}_{n\geq 1}$ is increasing and tending to $+\infty$, as $n\to\infty$. Thus, we can apply Stolz-Cesàro theorem and using formula (3.4), we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\left[H_{p,n+1}-H_{p,n}\right]\\
&= \frac{d^{1-p}}{p(1+p)^{(1-p)/p}}\cdot\lim_{n\to\infty}\frac{na_{n+1}^p-(a_1^p+\cdots+a_n^p)}{n^{p+1}}\\
&= \frac{d^{1-p}}{p(1+p)^{(1-p)/p}}\\
&\quad\times\lim_{n\to\infty}\frac{[(n+1)a_{n+2}^p-(a_1^p+\cdots+a_n^p+a_{n+1}^p)]-[na_{n+1}^p-(a_1^p+\cdots+a_n^p)]}{(n+1)^{p+1}-n^{p+1}}\\
&= \frac{d^{1-p}}{p(1+p)^{(1-p)/p}}\cdot\lim_{n\to\infty}\frac{(n+1)(a_{n+2}^p-a_{n+1}^p)}{(p+1)c_n^p},
\end{aligned}
$$

for some $c_n\in(n,\,n+1)$.

Finally, applying the Lagrange Mean Value theorem to the function $h(x)=x^p$ on the interval $[a_n,a_{n+1}]$, for each $n\geq N_2$, there exists $\eta_n\in(a_n,\,a_{n+1})$, such that

$$a_{n+1}^p-a_n^p=p\eta_n^{p-1}\left(a_{n+1}-a_n\right).$$

Thus, we obtain

$$\lim_{n\to\infty}\left[H_{p,n+1}-H_{p,n}\right]=\frac{d^{1-p}}{p(1+p)^{(1-p)/p}}\cdot\lim_{n\to\infty}\frac{(n+1)p\eta_n^{p-1}(a_{n+1}-a_n)}{(p+1)c_n^p}$$

$$= \frac{d^{1-p}}{(1+p)^{1/p}} \cdot \lim_{n\to\infty} \left\{ \frac{n+1}{c_n} \cdot \left[\frac{\eta_n}{c_n}\right]^{p-1} \cdot (a_{n+1} - a_n) \right\}$$
$$= \frac{d^{1-p}}{(1+p)^{1/p}} \cdot \lim_{n\to\infty} \frac{n+1}{c_n} \cdot \left[\lim_{n\to\infty} \frac{\eta_n}{c_n}\right]^{p-1} \cdot \lim_{n\to\infty} (a_{n+1} - a_n)$$
$$= \frac{d^{1-p}}{(1+p)^{1/p}} \cdot 1 \cdot d^{p-1} \cdot d = \frac{d}{(1+p)^{1/p}}.$$

Here, we used the fact that $\lim_{n\to\infty} \frac{\eta_n}{c_n} = d$, due to the fact that $a_n < \eta_n < a_{n+1}$ and $n < c_n < n+1$, and so we have

$$\frac{a_n}{n+1} < \frac{\eta_n}{c_n} < \frac{a_{n+1}}{n},$$

from which, since $a_n/n \to d$, as $n \to \infty$, we conclude that $\eta_n/c_n \to d$, as $n \to \infty$.

Case 3. If $p = -1$, then

$$H_{-1}(a_1, a_2, \ldots, a_n) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

is the harmonic mean of the numbers $a_1, a_2, \ldots, a_n$. Using the same short notation $H_{-1,n}$ for $H_{-1}(a_1, a_2, \ldots, a_n)$, for all $n \geq 1$, we have

(3.5)
$$H_{-1,n+1} - H_{-1,n} = \frac{n+1}{\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}}} - \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$
$$= \frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - \frac{n}{a_{n+1}}}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}}\right)}$$
$$= \frac{1}{\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}}}$$
$$- \frac{\frac{n}{a_{n+1}}}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}}\right)}.$$

Applying the Limit Comparison Test to the series with positive terms $\sum_{n=1}^{\infty} 1/a_n$ and $\sum_{n=1}^{\infty} 1/n$, since

$$\lim_{n\to\infty} \frac{1/a_n}{1/n} = \lim_{n\to\infty} \frac{n}{a_n} = \frac{1}{d} \in (0, \infty)$$

and the series $\sum_{n=1}^{\infty} 1/n$ is divergent, we conclude that the series $\sum_{n=1}^{\infty} 1/a_n$ is also divergent. Thus, using formula (3.5) we have

$$\lim_{n\to\infty} (H_{-1,n+1} - H_{-1,n}) = \lim_{n\to\infty} \frac{1}{\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}}}$$
$$- \frac{\lim_{n\to\infty} \frac{n}{a_{n+1}}}{\lim_{n\to\infty} \left[\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}}\right)\right]}$$
$$= \frac{1}{\infty} - \frac{1/d}{\infty \cdot \infty} = 0.$$

Moreover, using the Stolz-Cesàro theorem we have

$$\lim_{n\to\infty}\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{\ln(n)}=\lim_{n\to\infty}\frac{\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\frac{1}{a_{n+1}}\right)-\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)}{\ln(n+1)-\ln(n)}$$
$$=\lim_{n\to\infty}\frac{1}{\ln(1+(1/n))\cdot a_{n+1}}=\lim_{n\to\infty}\left[\frac{1}{n\ln(1+(1/n))}\right]\cdot\lim_{n\to\infty}\frac{n}{n+1}\cdot\lim_{n\to\infty}\frac{n+1}{a_{n+1}}$$
$$=\frac{1}{\lim_{n\to\infty}\ln\left((1+(1/n))^n\right)}\cdot 1\cdot\frac{1}{d}=\frac{1}{\ln(e)}\cdot\frac{1}{d}=\frac{1}{d}.$$

Hence, using formula (3.5), we have

$$\lim_{n\to\infty}\left[\ln(n)\cdot(H_{-1,n+1}-H_{-1,n})\right]$$
$$=\lim_{n\to\infty}\frac{\ln(n)}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\frac{1}{a_{n+1}}}\cdot\lim_{n\to\infty}\left[\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\frac{1}{a_{n+1}}\right)(H_{-1,n+1}-H_{-1,n})\right]$$
$$=d\cdot\lim_{n\to\infty}\left\{\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\frac{1}{a_{n+1}}\right)\cdot\left[\frac{1}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\frac{1}{a_{n+1}}}\right.\right.$$
$$\left.\left.-\frac{\frac{n}{a_{n+1}}}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\frac{1}{a_{n+1}}\right)}\right]\right\}$$
$$=d\cdot\lim_{n\to\infty}\left(1-\frac{\frac{n}{a_{n+1}}}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\right)=d\cdot\left(1-\frac{1/d}{\infty}\right)=d.$$

Thus, $H_{-1,n+1}-H_{-1,n}$ approaches 0 in the same the way that $d/\ln(n)$ goes to 0, as $n\to\infty$.

Case 4. If $-\infty<p<-1$, then let us define $q:=-p>1$. We have

(3.6)

$$H_{p,n+1}-H_{p,n}=\left(\frac{a_1^p+\cdots+a_n^p+a_{n+1}^p}{n+1}\right)^{1/p}-\left(\frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\right)^{1/p}$$
$$=\frac{(n+1)^{1/q}}{\left(\frac{1}{a_1^q}+\cdots+\frac{1}{a_n^q}+\frac{1}{a_{n+1}^q}\right)^{1/q}}-\frac{n^{1/q}}{\left(\frac{1}{a_1^q}+\frac{1}{a_2^q}+\cdots+\frac{1}{a_n^q}\right)^{1/q}}$$
$$=\frac{(n+1)^{1/q}\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}-n^{1/q}\left[\sum_{k=1}^{n+1}\frac{1}{a_k^q}\right]^{1/q}}{\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}\cdot\left[\sum_{k=1}^{n+1}\frac{1}{a_k^q}\right]^{1/q}}$$
$$=\frac{\left[(n+1)^{1/q}-n^{1/q}\right]\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}-n^{1/q}\left\{\left[\sum_{k=1}^{n+1}\frac{1}{a_k^q}\right]^{1/q}-\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}\right\}}{\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}\cdot\left[\sum_{k=1}^{n+1}\frac{1}{a_k^q}\right]^{1/q}}.$$

For each $n\geq 1$, applying Lagrange Mean Value theorem to the function $f(x)=x^{1/q}$ on the intervals $[n,n+1]$ and $[\sum_{k=1}^{n}\frac{1}{a_k^q},\sum_{k=1}^{n+1}\frac{1}{a_k^q}]$, there exist $c_n\in(n,n+1)$ and

$\alpha_n \in (\sum_{k=1}^{n} \frac{1}{a_k^q}, \sum_{k=1}^{n+1} \frac{1}{a_k^q})$, such that

$$(n+1)^{1/q} - n^{1/q} = \frac{1}{q} c_n^{(1-q)/q}$$

and

$$\left[\sum_{k=1}^{n+1} \frac{1}{a_k^q}\right]^{1/q} - \left[\sum_{k=1}^{n} \frac{1}{a_k^q}\right]^{1/q} = \frac{1}{q} \alpha_n^{(1-q)/q} \frac{1}{a_{n+1}^q}.$$

Substituting the last two formulas into formula (3.6), we obtain

(3.7)

$$\begin{aligned} H_{p,n+1} - H_{p,n} &= \frac{(1/q) c_n^{(1-q)/q} \left[\sum_{k=1}^{n} \frac{1}{a_k^q}\right]^{1/q} - n^{1/q} (1/q) \alpha_n^{(1-q)/q} \frac{1}{a_{n+1}^q}}{\left[\sum_{k=1}^{n} \frac{1}{a_k^q}\right]^{1/q} \left[\sum_{k=1}^{n+1} \frac{1}{a_k^q}\right]^{1/q}} \\ &= \frac{1}{q} \cdot \frac{c_n^{(1-q)/q} \left[\sum_{k=1}^{n} \frac{1}{a_k^q}\right]^{1/q} - n^{(1/q)-q} \alpha_n^{(1-q)/q} \frac{n^q}{a_{n+1}^q}}{\left[\sum_{k=1}^{n} \frac{1}{a_k^q}\right]^{1/q} \left[\sum_{k=1}^{n+1} \frac{1}{a_k^q}\right]^{1/q}}. \end{aligned}$$

Since we have

$$\lim_{n\to\infty} \frac{1/a_n^q}{1/n^q} = \left(\lim_{n\to\infty} \frac{n}{a_n}\right)^q = 1/d^q \in (0, \infty),$$

and the series $\sum_{n=1}^{\infty} (1/n^q)$ converges (because $q > 1$), $\sum_{n=1}^{\infty} (1/a_n^q)$ converges, too. Therefore, the sequence $\{1/a_n\}_{n\geq 1}$ belongs to the l^q-space.

Because for all $n \geq 1$, $n < c_n < n+1$, and $(1-q)/q < 0$, the Squeeze theorem implies that $c_n^{(1-q)/q} \to 0$, as $n \to \infty$. Also, because for all $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{a_k^q} < \alpha_n < \sum_{k=1}^{n+1} \frac{1}{a_k^q}$, the Squeeze theorem implies that $\alpha_n \to \sum_{k=1}^{\infty} \frac{1}{a_k^q} = \|\{1/a_n\}_{n\geq 1}\|_q^q$, as $n \to \infty$. Passing to the limit, as $n \to \infty$, in equation (3.7), since $(1/q) - q < (1/q) - 1 < 0$, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \left[H_p(a_1, \ldots, a_n, a_{n+1}) - H_p(a_1, a_2, \ldots, a_n)\right] \\ &= \frac{1}{q} \cdot \frac{1}{\lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{a_k^q}\right]^{1/q} \cdot \lim_{n\to\infty} \left[\sum_{k=1}^{n+1} \frac{1}{a_k^q}\right]^{1/q}} \\ &\times \left[\lim_{n\to\infty} c_n^{(1-q)/q} \cdot \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{a_k^q}\right]^{1/q} - \lim_{n\to\infty} n^{(1/q)-q} \cdot \lim_{n\to\infty} \alpha_n^{(1-q)/q} \cdot \lim_{n\to\infty} \frac{n^q}{a_{n+1}^q}\right] \\ &= \frac{1}{q} \cdot \frac{0 \cdot \left[\sum_{k=1}^{\infty} \frac{1}{a_k^q}\right]^{1/q} - 0 \cdot \left[\sum_{k=1}^{\infty} \frac{1}{a_k^q}\right]^{(1-q)/q} \cdot \frac{1}{d^q}}{\left[\sum_{k=1}^{\infty} \frac{1}{a_k^q}\right]^{2/q}} = 0. \end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left\{n^{1-(1/q)}\cdot\left[H_p\left(a_1,\ldots,a_n,a_{n+1}\right)-H_p\left(a_1,a_2,\ldots,a_n\right)\right]\right\}\\
&\quad=\lim_{n\to\infty}\left\{n^{1-(1/q)}\cdot\frac{1}{q}\cdot\frac{c_n^{(1-q)/q}\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}-n^{(1/q)-q}\alpha_n^{(1-q)/q}\frac{n^q}{a_{n+1}^q}}{\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}\left[\sum_{k=1}^{n+1}\frac{1}{a_k^q}\right]^{1/q}}\right\}\\
&\quad=\frac{1}{q}\cdot\lim_{n\to\infty}\left\{\frac{(c_n/n)^{(1-q)/q}\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}-n^{1-q}\alpha_n^{(1-q)/q}\frac{n^q}{a_{n+1}^q}}{\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}\left[\sum_{k=1}^{n+1}\frac{1}{a_k^q}\right]^{1/q}}\right\}
\end{aligned}$$

Each factor and term from the numerator and denominator of the last fraction has a finite limit as $n\to\infty$. Thus, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left\{n^{1-(1/q)}\cdot\left[H_{p,n+1}-H_{p,n}\right]\right\}\\
&=\frac{1}{q}\cdot\frac{1}{\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}\cdot\lim_{n\to\infty}\left[\sum_{k=1}^{n+1}\frac{1}{a_k^q}\right]^{1/q}}\\
&\times\left[\lim_{n\to\infty}(c_n/n)^{(1-q)/q}\cdot\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{a_k^q}\right]^{1/q}-\lim_{n\to\infty}n^{1-q}\cdot\lim_{n\to\infty}\alpha_n^{(1-q)/q}\cdot\lim_{n\to\infty}\frac{n^q}{a_{n+1}^q}\right]\\
&=\frac{1}{q}\cdot\frac{1\cdot\left[\sum_{k=1}^{\infty}\frac{1}{a_k^q}\right]^{1/q}-0\cdot\left[\sum_{k=1}^{\infty}\frac{1}{a_k^q}\right]^{(1-q)/q}\cdot\frac{1}{d^q}}{\left[\sum_{k=1}^{\infty}\frac{1}{a_k^q}\right]^{2/q}}=\frac{1}{q\|\{1/a_n\}_{n\ge1}\|_q},
\end{aligned}$$

since, for all $n\ge1$, we have $n/n<c_n/n<(n+1)/n$, and so $c_n/n\to1$, as $n\to\infty$.

Thus, $\{H_p(a_1,\ \ldots,\ a_n,\ a_{n+1})-H_p(a_1,\ a_2,\ \ldots,\ a_n)\}_{n\ge1}$ converges to 0 like $\{n^{(1/q)-1}\}_{n\ge1}$.

Case 5. If $p=-\infty$, then since $a_n\to\infty$, as $n\to\infty$, there exists $N\in\mathbb{N}$, such that $a_N=\min\{a_n\mid n\in\mathbb{N}\}$. Thus, for all $n\ge N$, we have

$$H_{-\infty}\left(a_1,\ldots,a_n,a_{n+1}\right)-H_{-\infty}\left(a_1,a_2,\ldots,a_n\right)=a_N-a_N=0.$$

Therefore, we have

$$\lim_{n\to\infty}\left[H_{-\infty}\left(a_1,\ldots,a_n,a_{n+1}\right)-H_{-\infty}\left(a_1,a_2,\ldots,a_n\right)\right]=0.\quad\blacksquare$$

Corollary 3.2. *Let $\{a_n\}_{n\ge1}$ be a sequence of positive numbers such that the sequence $\{a_{n+1}-a_n\}_{n\ge1}$ is convergent and $d:=\lim_{n\to\infty}(a_{n+1}-a_n)>0$. Then,*

- *For $-1<p\le\infty$, we have*

$$\lim_{n\to\infty}\frac{H_p(a_1,a_2,\ldots,a_n)}{n}=\frac{d}{(1+p)^{1/p}},$$

where for $p = \infty$, $(1+p)^{1/p} := \lim_{r\to\infty}(1+r)^{1/r} = 1$, *and for* $p = 0$, $(1+p)^{1/p} := \lim_{r\to 0}(1+r)^{1/r} = e$.

- *For* $p = -1$, *we have*

$$\lim_{n\to\infty} \frac{\ln(n) \cdot H_{-1}(a_1, a_2, \ldots, a_n)}{n} = d.$$

- *For* $-\infty < p < -1$, *defining* $q := -p$, *we have*

$$\lim_{n\to\infty} \{n^{1/p} \cdot H_p(a_1, a_2, \ldots, a_n)\} = \frac{1}{\|\{1/a_n\}\|_q}.$$

Proof. Indeed, using the Stolz-Cesàro theorem and the previous theorem:

- If $p > -1$, then we have

$$\lim_{n\to\infty} \frac{H_p(a_1, a_2, \ldots, a_n)}{n} = \lim_{n\to\infty} \frac{H_p(a_1, \ldots, a_n, a_{n+1}) - H_p(a_1, a_2, \ldots, a_n)}{n+1-n}$$

$$= \lim_{n\to\infty} [H_p(a_1, \ldots, a_n, a_{n+1}) - H_p(a_1, a_2, \ldots, a_n)] = \frac{d}{(1+p)^{1/p}}.$$

- If $p = -1$, then we have

$$\lim_{n\to\infty} \frac{H_p(a_1, a_2, \ldots, a_n)}{n/\ln(n)} = \lim_{n\to\infty} \frac{H_p(a_1, \ldots, a_n, a_{n+1}) - H_p(a_1, a_2, \ldots, a_n)}{(n+1)/\ln(n+1) - n/\ln(n)},$$

where here, in order to be allowed to apply the Stolz-Cesàro theorem, we need to check that the sequence $\{n/\ln(n)\}_{n\geq N}$, is strictly increasing for some large N. Indeed, the function $f : (1, \infty) \to \mathbb{R}$, defined by $f(x) = x/\ln(x)$, has the derivative

$$f'(x) = \frac{1 \cdot \ln(x) - x \cdot (1/x)}{\ln^2(x)} = \frac{\ln(x) - 1}{\ln^2(x)} > 0,$$

for all $x \in (e, \infty)$. Thus, we can see that $\{n/\ln(n)\}_{n\geq 3}$ is strictly increasing.

Applying the Lagrange Mean Value theorem to the function $f(x) = x/\ln(x)$, on each interval $[n, n+1]$, for $n \geq 3$, there exists $r_n \in (n, n+1)$, such that

$$\frac{n+1}{\ln(n+1)} - \frac{n}{\ln(n)} = \frac{\ln(r_n) - 1}{\ln^2(r_n)}.$$

Thus, we have

$$\lim_{n\to\infty} \frac{H_p(a_1, a_2, \ldots, a_n)}{n/\ln(n)} = \lim_{n\to\infty} \frac{H_p(a_1, \ldots, a_n, a_{n+1}) - H_p(a_1, a_2, \ldots, a_n)}{(n+1)/\ln(n+1) - n/ln(n)}$$

$$= \lim_{n\to\infty} \frac{\ln^2(r_n)[H_p(a_1, \ldots, a_n, a_{n+1}) - H_p(a_1, a_2, \ldots, a_n)]}{\ln(r_n) - 1}$$

$$= \lim_{n\to\infty} \{\ln(n) \cdot [H_p(a_1, \ldots, a_n, a_{n+1}) - H_p(a_1, a_2, \ldots, a_n)]\}$$

$$\times \lim_{n\to\infty} \frac{\ln(r_n)}{\ln(n)} \cdot \lim_{n\to\infty} \frac{\ln(r_n)}{\ln(r_n) - 1} = d \cdot 1 \cdot 1 = d,$$

since $n < r_n < n+1$ for all $n \geq 3$.

- If $-\infty < p < -1$, then denoting $q := -p > 1$, and using the Stolz-Cesàro theorem, we have

$$\lim_{n\to\infty}\left[n^{1/p}\cdot H_p(a_1,a_2,\dots,a_n)\right] = \lim_{n\to\infty}\frac{H_p(a_1,a_2,\dots,a_n)}{n^{1/q}}$$
$$= \lim_{n\to\infty}\frac{H_p(a_1,\dots,a_n,a_{n+1}) - H_p(a_1,a_2,\dots,a_n)}{(n+1)^{1/q}-n^{1/q}}.$$

Applying the Lagrange Mean Value theorem to $f(x) = x^{1/q}$ on each interval $[n, n+1]$, for all $n \in \mathbb{N}$, there exists $c_n \in (n,\, n+1)$, such that

$$(n+1)^{1/q} - n^{1/q} = \frac{1}{q}c_n^{(1/q)-1}(n+1-n) = \frac{1}{q}c_n^{-(1/p)-1}.$$

Thus, using our theorem and the fact that $c_n/n \to 1$, as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\left[n^{1/p}\cdot H_p(a_1,a_2,\dots,a_n)\right]$$
$$= \lim_{n\to\infty}\frac{H_p(a_1,\dots,a_n,a_{n+1}) - H_p(a_1,a_2,\dots,a_n)}{(n+1)^{1/q}-n^{1/q}}$$
$$= \lim_{n\to\infty}\frac{H_p(a_1,\dots,a_n,a_{n+1}) - H_p(a_1,a_2,\dots,a_n)}{(1/q)c_n^{-(1/p)-1}}$$
$$= q\lim_{n\to\infty}\left\{n^{\frac{1}{p}+1}\left[H_p(a_1,\dots,a_n,a_{n+1}) - H_p(a_1,a_2,\dots,a_n)\right]\right\}\cdot\lim_{n\to\infty}\left(\frac{c_n}{n}\right)^{\frac{1}{p}+1}$$
$$= q\cdot\frac{1}{q\|\{1/a_n\}_{n\ge1}\|_q}\cdot 1^{\frac{1}{p}+1} = \frac{1}{\|\{1/a_n\}_{n\ge1}\|_q}. \quad \blacksquare$$

REFERENCES

[1] D. M. Bătineţu-Giurgiu, D. Sitaru, N. Stanciu, *Two classes of Lalescu's sequences*, Octogon Mathematical Magazine, **27**, 2, (2019), 805–813.

[2] W. G. Boskoff, B. D. Suceavă, *A class of applications of AM-GM Inequality: From a 2004 Putnam competition problem to Lalescu's sequence*, Gazette, **5**, 32 (2005), 51–56.

[3] T. Lalescu, *Problem 579*, Mathematical Gazzete, **6** (1900-1901), 148.

[4] G. Pólya, G. Szegö, *Problems and Theorems in Analysis I*, Springer-Verlag, Berlin Heidelberg New York, 1978.

[5] H. J. Schultz, *The sum of the kth power of the first n integers*, Amer. Math. Monthly, **87** (1980), 478–481.

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