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RELAXATION OF THE SEQUENTIAL CRITERIA FOR CONTINUITY AND UNIFORM CONTINUITY OF A REAL FUNCTION

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Abstract. We show how two fundamental sequential criteria for real functions, namely the sequential criterion for continuity and the sequential criterion for uniform continuity, are relaxed. Further, we derive a global continuity criterion, as an immediate consequence of the relaxed sequential criterion for continuity, and a fundamental property of uniformly continuous functions, as an application of the relaxed sequential criterion for uniform continuity.

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1. Introduction

The foundations of real analysis were laid in the nineteenth century by Bolzano, Cauchy, Weierstrass, Heine, Riemann and others. It was in that period that fundamental concepts, such as the limit and the continuity of a real function, were rigorously defined and the known ε-δ definitions were introduced [4, 6]. Then it became known that each such ε-δ definition can be expressed as an equivalent sequential criterion in the sense that the ε-δ definition implies the sequential criterion and the converse [1, 2, 3, 7, 8].

For instance, the ε-δ definition of limit of a real function at a cluster point of its domain is equivalent[1] to a sequential criterion, known as the sequential criterion for limits, which expresses the function limit in terms of the limit of appropriately chosen sequences of function values and thus it allows for the function limit to be studied as a sequence limit. In this way, properties of function limits can be directly derived from respective properties of sequence limits.

In this work, we focus on the sequential criteria for continuity and uniform continuity, and we show how they can be relaxed, as a consequence of the presence

[1]It should be noted that Heine used this criterion as the (sequential) definition of limit of a real function and although the ε-δ definition implies straightforwardly Heine's definition, for Heine's definition to imply the ε-δ definition, the Axiom of Choice is needed (see [5, pp. 145–146], where the necessity of the Axiom of Choice is demonstrated in the similar case of the sequential criterion for continuity, which was used also by Heine as the (sequential) definition of continuity).

of the universal quantifier in their statements. We also derive a global continuity criterion, as an immediate consequence of the relaxed sequential criterion for continuity of a real function on $\mathbb{R}$, and a fundamental property of uniformly continuous functions, as an application of the relaxed sequential criterion for uniform continuity.

Taking into account that the relevant discussion can be rarely seen in standard analysis textbooks, the present work could be hopefully used to fill in this gap in the literature.

In what follows, whenever the notation is not standard, we follow [2].

2. The sequential criterion for continuity

As a result of the ε-δ definition of continuity of a real function at a point of its domain, the following sequential criterion holds [1, 2, 7].

Theorem 1 (Sequential criterion for continuity). *Let $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in A$. f is continuous at x_0 if and only if for every sequence (x_n) in A, if (x_n) converges to x_0, then the sequence $(f(x_n))$ of function values converges to $f(x_0)$.*

In order to relax the sequential criterion for continuity, we will prove the following lemma.

Lemma 1. *Let $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let x_0 be a cluster point of A. If for every sequence (x_n) in A, (if (x_n) converges to x_0, then the sequence $(f(x_n))$ converges), then $(f(x_n))$ converges to $f(x_0)$.*

Proof. We assume that for every sequence (x_n) in A, if (x_n) converges to x_0, then the sequence $(f(x_n))$ converges.

First, we will show that all such sequences $(f(x_n))$ converge to the same limit. Aiming at a contradiction, we assume that this is not the case. Then there exist two sequences (x_n) and (y_n) in A that both converge to x_0 and the respective sequences $(f(x_n))$ and $(f(y_n))$ converge, but to different limits.

Next, we consider the sequence (z_n) defined by $z_{2n-1} = x_n$ and $z_{2n} = y_n$, for all $n \in \mathbb{N}$. Since both sequences (x_n) and (y_n) are in A, then, clearly, the sequence (z_n) is in A, too. Further, since both sequences (x_n) and (y_n) converge to x_0, then both subsequences (z_{2n-1}) and (z_{2n}) of (z_n) converge to x_0, and as a consequence, the sequence (z_n) itself converges to x_0, too.

Then, by assumption, the sequence $(f(z_n))$ converges and as a consequence, the subsequences $(f(z_{2n-1})) = (f(x_n))$ and $(f(z_{2n})) = (f(y_n))$ of $(f(z_n))$ also converge and to the same limit as $(f(z_n))$. Consequently, the sequences $(f(x_n))$ and $(f(y_n))$ converge to the same limit, which is a contradiction. This establishes the statement that if for every sequence (x_n) in A, (if (x_n) converges to x_0, then the sequence $(f(x_n))$ converges), then $(f(x_n))$ always converges to the same limit.

Next, we will show, by direct proof, that the previous common limit is equal to $f(x_0)$. To this end, we consider the sequence (x_n) defined by $x_n = x_0$ for all

$n \in \mathbb{N}$. Since (x_n) is a constant sequence with value x_0, then it converges to x_0. Also, since $x_0 \in A$, then (x_n) is in A. Then, by the statement we have just proved, the sequence $(f(x_n))$ converges to the previous common limit and since $(f(x_n))$ is a constant sequence with value $f(x_0)$, then it converges to $f(x_0)$, which is then the previous common limit. ∎

THEOREM 2 (RELAXED SEQUENTIAL CRITERION FOR CONTINUITY). *Let $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in A$ be a cluster point of A. f is continuous at x_0 if and only if for every sequence (x_n) in A, if (x_n) converges to x_0, then the sequence $(f(x_n))$ converges.*

Proof. (i) Let f be continuous at x_0. Then, by the sequential criterion for continuity, for every sequence (x_n) in A, if (x_n) converges to x_0, then the sequence $(f(x_n))$ converges to $f(x_0)$; hence it converges.

(ii) We assume that for every sequence (x_n) in A, if (x_n) converges to x_0, then the sequence $(f(x_n))$ converges. Then, by Lemma 1, $(f(x_n))$ converges to $f(x_0)$ and thus, by the sequential criterion for continuity, f is continuous at x_0. ∎

REMARK 1. Invoking the Cauchy convergence criterion, we can replace the phrase "the sequence $(f(x_n))$ converges" in the statement of the previous relaxed criterion with the phrase "the sequence $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$".

The following global continuity criterion is an immediate consequence of the relaxed sequential criterion for continuity.

THEOREM 3 (GLOBAL CONTINUITY CRITERION). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is continuous on $\mathbb{R}$ if and only if for every real sequence (x_n), if (x_n) converges, then the sequence $(f(x_n))$ converges. In other words, a real function is continuous on $\mathbb{R}$ if and only if it sends real Cauchy sequences to real Cauchy sequences.*

Proof. (i) Let f be continuous on $\mathbb{R}$ and let (x_n) be any convergent real sequence. Then there exists $x_0 \in \mathbb{R}$ such that (x_n) converges to x_0. Since f is continuous on $\mathbb{R}$ and $x_0 \in \mathbb{R}$, then f is continuous at x_0. Then, by the relaxed sequential criterion for continuity, $(f(x_n))$ converges.

(ii) We assume that for every real sequence (x_n), if (x_n) converges, then $(f(x_n))$ also converges. Let any $x_0 \in \mathbb{R}$ be given and let (x_n) be any real sequence that converges to x_0. Then, by assumption, the sequence $(f(x_n))$ converges. Hence, by the relaxed sequential criterion for continuity, f is continuous at x_0, and since x_0 is any real number, then f is continuous on $\mathbb{R}$. ∎

3. The sequential criterion for uniform continuity

As a result of the ε-δ definition of uniform continuity of a real function on its domain, the following sequential criterion holds [1, 2][2].

[2]In both references, the negation of the statement of the present criterion is proved and is referred to as non-uniform continuity criterion.

Theorem 4 (Sequential criterion for uniform continuity). *Let $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. f is uniformly continuous on A if and only if for all sequences (x_n) and (y_n) in A, if the sequence $(x_n - y_n)$ converges to zero, then the sequence $(f(x_n) - f(y_n))$ also converges to zero.*

As in the case of continuity, the ε-δ definition of uniform continuity implies straightforwardly the sequential criterion for uniform continuity, but for the converse to hold, the Axiom of Choice must be invoked.

In order to relax the sequential criterion for uniform continuity, we will prove the following lemma.

Lemma 2. *Let $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. If for all sequences (x_n) and (y_n) in A, (if $(x_n - y_n)$ converges to zero, then $(f(x_n) - f(y_n))$ converges), then $(f(x_n) - f(y_n))$ converges to zero.*

Proof. We assume that for all sequences (x_n) and (y_n) in A, if $(x_n - y_n)$ converges to zero, then $(f(x_n) - f(y_n))$ converges.

First, we will show that all such sequences $(f(x_n) - f(y_n))$ converge to the same limit. Aiming at a contradiction, we assume that not all such sequences $(f(x_n) - f(y_n))$ converge to the same limit. Then there exist sequences (x_n) and (y_n), and (a_n) and (b_n), all in A, such that $(x_n - y_n)$ converges to zero and $(a_n - b_n)$ converges to zero, and such that the sequences $(f(x_n) - f(y_n))$ and $(f(a_n) - f(b_n))$ both converge, but to different limits.

Next, we consider the sequence (s_n) defined by $s_{2n-1} = x_n$ and $s_{2n} = a_n$, for all $n \in \mathbb{N}$, and the sequence (t_n) defined by $t_{2n-1} = y_n$ and $t_{2n} = b_n$, for all $n \in \mathbb{N}$. Since the sequences (x_n) and (a_n) are both in A, then the subsequences (s_{2n-1}) and (s_{2n}) of (s_n) are both in A; whence the sequence (s_n) is in A, too. In the same way, the sequence (t_n) is in A, too. Further, we observe that, for every $n \in \mathbb{N}$, $s_{2n-1} - t_{2n-1} = x_n - y_n$, and since $(x_n - y_n)$ converges to zero, then the subsequence $(s_{2n-1} - t_{2n-1})$ of $(s_n - t_n)$ converges to zero. Similarly, for every $n \in \mathbb{N}$, $s_{2n} - t_{2n} = a_n - b_n$, and since $(a_n - b_n)$ converges to zero, then the subsequence $(s_{2n} - t_{2n})$ of $(s_n - t_n)$ converges to zero. As a consequence, the sequence $(s_n - t_n)$ itself converges to zero, too.

We have thus shown that the sequences (s_n) and (t_n) are both in A, and their difference $(s_n - t_n)$ converges to zero. Then, by assumption, the sequence $(f(s_n) - f(t_n))$ converges. Consequently, the subsequences $(f(s_{2n-1}) - f(t_{2n-1}))$ and $(f(s_{2n}) - f(t_{2n}))$ of $(f(s_n) - f(t_n))$ converge to the same limit; i.e., the sequences $(f(x_n) - f(y_n))$ and $(f(a_n) - f(b_n))$ converge to the same limit, which is a contradiction.

We have thus proved that if for all sequences (x_n) and (y_n) in A, (if $(x_n - y_n)$ converges to zero, then $(f(x_n) - f(y_n))$ converges), then all such sequences $(f(x_n) - f(y_n))$ converge to the same limit.

Next, we will show, by direct proof, that the previous common limit is equal to zero. To this end, we consider any sequence (x_n) in A and we choose the sequence

(y_n) such that $y_n = x_n$ for all $n \in \mathbb{N}$. Then (y_n) is in A, too, and since $x_n - y_n = 0$ for all $n \in \mathbb{N}$, then the sequence $(x_n - y_n)$ converges to zero. Further, for all $n \in \mathbb{N}$, $f(x_n) - f(y_n) = 0$; whence the sequence $(f(x_n) - f(y_n))$ also converges to zero. Also, by the statement we have just proved, $(f(x_n) - f(y_n))$ converges to the common limit we are looking for. Hence the common limit is zero. ∎

Theorem 5 (Relaxed sequential criterion for uniform continuity). *Let $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. f is uniformly continuous on A if and only if for all sequences (x_n) and (y_n) in A, if the sequence $(x_n - y_n)$ converges to zero, then the sequence $(f(x_n) - f(y_n))$ converges.*

Proof. (i) Let f be uniformly continuous on A. Then, by the sequential criterion for uniform continuity, for all sequences (x_n) and (y_n) in A, if the sequence $(x_n - y_n)$ converges to zero, then the sequence $(f(x_n) - f(y_n))$ also converges to zero; hence it converges.

(ii) We assume that for all sequences (x_n) and (y_n) in A, if the sequence $(x_n - y_n)$ converges to zero, then the sequence $(f(x_n) - f(y_n))$ converges. Then, by Lemma 2, $(f(x_n) - f(y_n))$ converges to zero and thus, by the sequential criterion for uniform continuity, f is uniformly continuous on A. ∎

Remark 2. Invoking the Cauchy convergence criterion, we can replace the phrase "the sequence $(f(x_n) - f(y_n))$ converges" in the statement of the previous relaxed criterion with the phrase "the sequence $(f(x_n) - f(y_n))$ is a Cauchy sequence in $\mathbb{R}$".

As an application of the relaxed sequential criterion for uniform continuity, we will show that a uniformly continuous function sends Cauchy sequences in its domain to Cauchy sequences in $\mathbb{R}$. To this end, we will need the following lemma.

Lemma 3. *Let (x_n) be a real sequence. If (x_n) diverges, then there exist two subsequences of (x_n) whose difference also diverges.*

Proof. Let (x_n) diverge. Then (x_n) may be bounded or unbounded. We will distinguish the two cases.

(i) If (x_n) is bounded, then, by the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence (x_{n_k}). Clearly, (x_n) is a subsequence of itself. Then the sequence $(x_k - x_{n_k})$ is the difference of two subsequences of (x_n) and diverges; otherwise both sequences $(x_k - x_{n_k})$ and (x_{n_k}) converge, and then the sequence $\big((x_k - x_{n_k}) + x_{n_k}\big) = (x_k)$ converges, which is a contradiction.

(ii) If (x_n) is unbounded, then it is unbounded above or unbounded below. Since the two cases are similar, we will examine only the first case. Since (x_n) is unbounded above, then there exists a term x_{n_1} of (x_n) such that $x_{n_1} > x_1 + 1$.

Since (x_n) is unbounded above, then every tail of (x_n) is unbounded above; otherwise all but finitely many terms of (x_n) are bounded above by some $M \in \mathbb{R}$ and since there exists a maximum term in all those finitely many terms, then the maximum of that maximum term and M is an upper bound of (x_n), which is a contradiction, since (x_n) is unbounded above.

Since the n_1-tail of (x_n) is unbounded above, then there exists a term x_{n_2} of (x_n), with $n_2 > n_1$, such that $x_{n_2} > x_2 + 2$. Continuing in the same way, we construct a strictly increasing sequence (n_k) of natural numbers such that $x_{n_k} > x_k + k$, for all $k \in \mathbb{N}$. Then (x_{n_k}) is a subsequence of (x_n) (which is a subsequence of itself) and $x_{n_k} - x_k > k$ for all $k \in \mathbb{N}$; whence the difference $(x_{n_k} - x_k)$ diverges to positive infinity; thus it diverges. ■

THEOREM 6. *If a function $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$ is uniformly continuous on A, then for every Cauchy sequence (x_n) in A, $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.*

Proof. Let f be uniformly continuous on A and let (x_n) be a Cauchy sequence in A. We consider the difference $(f(x_{n_k}) - f(x_{m_k}))$ of any two subsequences $(f(x_{n_k}))$ and $(f(x_{m_k}))$ of the sequence $(f(x_n))$. Clearly, (x_{n_k}) and (x_{m_k}) are two subsequences of (x_n). Since (x_n) is a Cauchy sequence, then, by the Cauchy convergence criterion, it converges in $\mathbb{R}$ (but not necessarily to a point of A). Hence, the subsequences (x_{n_k}) and (x_{m_k}) of (x_n) both converge and to the same limit as (x_n). Consequently, the difference $(x_{n_k} - x_{m_k})$ converges to zero. Also, since (x_n) is in A, then both (x_{n_k}) and (x_{m_k}) are in A. Then[3], by the relaxed sequential criterion for uniform continuity, $(f(x_{n_k}) - f(x_{m_k}))$ converges. Then, by the contrapositive of Lemma 3, $(f(x_n))$ converges and thus is a Cauchy sequence (in $\mathbb{R}$), which is what we wanted to prove. ■

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[3]Any subsequence of a given sequence is a sequence in its own right.