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TOWARDS NONLINEAR EQUATIONS – A CASE TEACHING APPROACH IN MATHEMATICAL ANALYSIS OF FUNCTIONS OF ONE VARIABLE

Marek Galewski and Jakub Łompieś

Abstract. In this note we are concerned with the proposal of case study approach in teaching mathematical analysis. By describing a simple student's project about the solvability of (nonlinear) equations we aim at indicating that many possible further directions are also possible.

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1. Introduction

Teaching of mathematical analysis of functions of one variable relies on utmost precision, understanding of theorems and numerous notions, acquainting with calculations and problem solving skills all devised for various parts of the course, see the excellent textbook [17] which will be unrivaled for many years to come. Instructors usually indicate where the methods introduced will be required in the future study and which notions are of some importance. The main issue observed by teachers is that very often problem solving skills lead to more advanced problem solving skills and at the end of a course or a chapter, no overall conclusions are being drawn, as observed in diverse books containing problems, like [1, 4]. Usually, especially in Polish universities, selected methods of mathematical analysis are taught one by one with emphasis on precision, with all required proofs and with assumptions being the weakest possible, however nothing is summarized about them in the end. The problems which are assigned to students are often focused on application of some method that has just been introduced. Can the beginners already approach a sort of case teaching method focused from the very beginning on the outline of the theory which is being taught and being concluded with the overall summary depicting most important issues as is discussed in [7, 11]? Yes, they do. We shall show it by example of the following assignment project:

(Q) *Which theorems of mathematical analysis (meaning the analysis of functions of one variable) lead to the existence of solutions to nonlinear equations?*

Being more precise this means as follows:

Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ *and equation*

$$f(x) = 0. \tag{1}$$

Provide, upon suitable assumptions on f (depending on the method chosen) conditions leading to the existence of solutions to problem (1).

Such a project allows us to summarize what have been grasped during the course and moreover, it allows us to describe many methods that lie behind the theorems.

This note is organized as follows. Firstly, in Section 2, we give some mathematical background which is given to make clear what is implied by the further performed case solution and next provide an answer to such posed question from student's perspective (by the second author). Section 3 is devoted to description of theoretical methods accessible to first year students which help in solving problem (Q) (by the first author). These methods are mentioned, commented and also compared.

2. Student's case solution

The following section contains the case study solution to the question (Q) and problem (1) as performed by the students. Out of many projects received, we decided to chose the project which approaches the lead question in a most suitable way. It was unsupervised, meaning that the choice of methods and the example was done by the student. Nevertheless the mathematical content as well as the solutions were checked for consistency, correctness and compliance with the main problem.

2.1. Theoretical background

The theoretical background used in this article is provided by the following textbooks [2, 3, 6, 13, 17]. Assertions such as the Fermat rule, Rolle theorem, the minimization principle, the inversion of a monotone and continuous function, Fundamental Theorem of Algebra, Little Bézout Theorem, Sturm theory will come to our help in treating the problem under consideration. We also have to mention the Intermediate Value Theorem for continuous functions that implies further the existence of zeros to f which we cite due to its importance in what follows:

Theorem 1 [Existence of zeros] *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. If $f(a)f(b) < 0$ then there is $x_0 \in (a,b)$ such that $f(x_0) = 0$.*

Remark 1. Additionally assuming that f is differentiable, if the sign of f' is constant on (a,b) then the zero of f over $[a,b]$ is unique.

Remark 2. Note also that the above result has hidden in it the geometric boundary conditions and compactness as well, apart from the continuity and connectedness. It appears that such concepts like the connectedness, the compactness, the continuity and the geometric conditions, understood possibly from some different point of view, will prove to be indispensable in demonstrating the existence of solutions to nonlinear equations, see for example [9] and [12].

When the existence of all roots has been proven we can find their approximate values by using one of those methods which come from the proofs of the existence theorems and which are still being developed by various numerical approaches. We provide these which we use further on.

The bisection method, which is always convergent (albeit slowly in many cases) is based on the existence of zeros.

The Newton method is based on replacing f with the tangent linear function whose zeros are approximated. The next approximations are the zeros of these functions. Unfortunately, Newton's method is not always convergent, especially if the starting point is too far away from the desired solution.

2.2. Answer to the problem (1)

Let f be continuous and the domain of f be an open interval $D_f := (a, b)$ where $-\infty \le a < b \le +\infty$.

1. Using available theorems or methods, we test whether the roots of the equation $f(x) = 0$ exist:
   - We start with calculating $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$.

     If they have different signs, we can prove the existence of at least one root using Intermediate Value Theorem.

     If $(\lim\limits_{x\to a^+} f(x) > 0$ and $\lim\limits_{x\to b^-} f(x) > 0)$, then we have to check whether $\inf\limits_{x\in D_f} f(x) \le 0$. In case when $(\lim\limits_{x\to a^+} f(x) < 0$ and $\lim\limits_{x\to b^-} f(x) < 0)$ we need to check if $\sup\limits_{x\in D_f} f(x) \ge 0$. *Additionally (if f is differentiable) we can use monotonicity and local extrema to draw the above conclusion.*
   - If f is differentiable, then we can use monotonicity and local extrema to conjecture about the amount of solutions. *We have to be cautious using monotonicity and local extrema when determining the existence of roots of the equation* (1). *For nonlinear functions (excluding polynomials) it does not always yield the desired result, consider* $f(x) := \frac{1}{2}x + \sin(x)$.
   - If f is a polynomial, then we can use Sturm theorem to find the exact amount of roots.
   - After that we use the existence of zeros theorem to find specific intervals where solutions are located. Moreover, we need to show that there are no other roots of the equation.

2. Knowing that solutions exist we have to choose and apply methods of approximate equation solving remembering about their additional assumptions:
   - Bisection method is good enough for any continuous f although as mentioned before it is rather "slow".
   - We apply Newton's method when $f \in C^2[a, b]$, in case where both a and b are finite, the first and second derivative of f have a constant sign on the considered interval and the starting point x_0 meets the requirement $f(x_0)f''(x_0) > 0$.

EXAMPLE 4. Find the real roots of the equation

$$f(x) := x^5 - 10x - 1 = 0$$

Function f is a polynomial of odd degree, so it is continuous and has at least one real root and at most five. Now we find the exact amount of solutions.

Method 1. We use monotonicity and local extrema to show that the desired solutions are in $(-\infty, -\sqrt[4]{2}), (-\sqrt[4]{2}, \sqrt[4]{2})$ and $(\sqrt[4]{2}, \infty)$, respectively.

Method 2. We apply Sturm's theory on the segment $[-4, 4]$ to show that three real roots exist.

Table 1: Application of Sturm's theory

Sturm's chain	$a = (-4)$	$b = 4$
$p_0(x) = f(x) = x^5 - 10x - 1$	-985	983
$p_1(x) = f'(x) = 5x^4 - 10$	1270	1270
$p_2(x) = 8x - 1$	-33	31
$p_3(x) = \frac{319}{320}$	$\frac{319}{320}$	$\frac{319}{320}$
$Z(x)$	3	0

Next, we prove that there are no more solutions. We use the existence of zeros theorem to show that the roots exist on intervals $[-2, -1]$, $[-1, 0]$ and $[1, 2]$ which allows us to apply approximation methods. The roots are given as follows:

$$r_1 \approx (-1{,}75)\,, \quad r_2 \approx (-0{,}1100001)\,, \quad r_3 \approx 1{,}8046875.$$

The first solution is obtained (after third iteration) on $[-2, -1]$ by the Newton's method. The second is found (after third iteration) on [-1,0] by the iteration method which is based on the Banach fixed point theorem, see [16] for a recent overview of the usage of fixed point methods. Finally, the third (after seventh iteration) on $[1, 2]$ by the bisection method.

We summarize our results concerning this example in Fig. 1 (on the next page).

3. Ideal solution – towards a new teaching paradigm

When one assigns a project, there is always a model solution that was at least sketched, see for example [10]. The case described in Section 2 is based on answers given by students towards the proposed project (Q) and it focuses rather on methods leading to finding solutions (either exact or approximate) and not that much on tools. At the same time, several existence tools are mentioned and further employed with success in the example. Therefore one may assert that the project is consistent, corresponds to the given question and provides detailed and correct answers. However some more experienced mathematicians clearly see

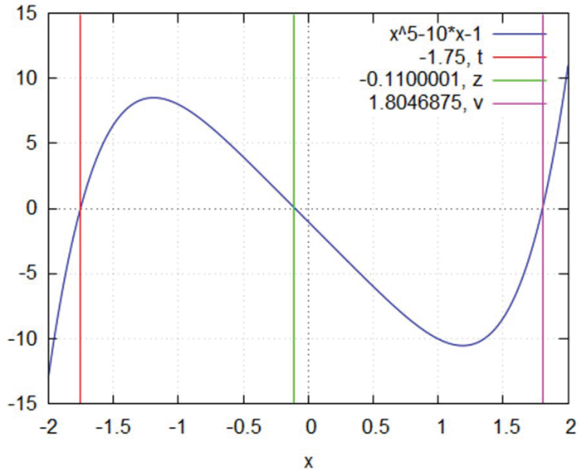

Figure 1. The graphical illustration

that several issues leading to the existence of solutions have not been addressed. Such composition of the project is not surprising since teaching focuses more on procedures than on overall understanding of methods. The obtained projects have great value in them, but concerns rather different question, namely: *What are the methods used to solve equation* (1), while our project was about question (Q). We would like to underline here that the theoretical methods contained in Section 2 were chosen accordingly to their further application in the example. One may ask why the existence is that important.

By asking such a question the following is implicitly put (after I. Ekeland in [5]): *What does it mean to solve an equation?* We cannot exactly solve even simple linear equation like

$$\pi \cdot x = e \tag{2}$$

about whose solution we know that it exists, is unique, depends continuously on parameters – i.e. it can be successfully numerically approximated. We can distinguish between existence methods involving differentiation and those which rely only on the continuity assumption about function f. Among the former we see that the following can be applied:

a1) f is continuous over $[a, b]$ (i.e. connectedness and compactness condition), $f(a) \cdot f(b) < 0$ (i.e. geometric boundary conditions); then the theorem about the Darboux property allows us to find $c \in (a, b)$ such that $f(c) = 0$;

b1) f is continuous on $\mathbb{R}$, either nonincreasing or nondecreasing (geometry); $\lim_{x\to\pm\infty} |f(x)| = +\infty$ (which is related to compactness); then we can find a closed interval (possibly degenerated) $C \subset \mathbb{R}$ such that $f(c) = 0$ for $c \in C$, see Proposition 1.1 in [8]; we see here that continuity together with the coercivity of $|f|$ provide that the set of solutions is non-empty, while the monotonicity is responsible for localisation.

Especially, the method described in point b1) is of some immense importance since it gives rise to the method of monotone operators, see [8], applied in various areas of engineering, for example in contact mechanics, see [15]. While in the whole generality it is much more involved, the first step is done here and can be further generalized. Uniqueness in case b1) is reached when f is strictly monotone.

In order to understand how to apply differentiation in the search of existence results, one may rewrite equation

$$f(x) = 0$$

in case when $f\colon \mathbb{R} \to \mathbb{R}$ is continuous as follows:

$$F'(x) = 0,$$

where the function $F\colon \mathbb{R} \to \mathbb{R}$ (namely the antiderivative) is defined as follows

$$F(x) = \int_0^x f(s)\, ds.$$

In such a case also the compactness, connectedness and geometry play their roles as described below (matching the above described observations a1) and b1)):

a2) F is differentiable over (a, b) and continuous over $[a, b]$ (i.e. compactness, connectedness and the regularity conditions), $F(a) = F(b) = 0$ (i.e. geometric boundary conditions); then Rolle Theorem applies and there is some $c \in (a, b)$ such that $F'(c) = f(c) = 0$;

b2) F is coercive, i.e. $\lim_{x\to\pm\infty} F(x) = +\infty$ or anti-coercive, i.e. $\lim_{x\to\pm\infty} F(x) = -\infty$, which implies that F has a global minimum (or maximum); then Fermat Rule applies and there is some $c \in \mathbb{R}$ such that $F'(c) = f(c) = 0$;

c2) F has an inflection point at some $c \in \mathbb{R}$; then also $f(c) = 0$.

Again when F is convex, the set of solutions to equation $f(x) = 0$ is a closed interval, since in this case the derivative of F, i.e. f is nondecreasing and a1) applies. The strict convexity renders the uniqueness by similar arguments. Such an approach as described here stands for the beginners guide to the calculus of variation, see [9, 12, 14] as a few references in the area. This world of variational methods is very rich with useful results. One can easily draw what the conclusion is when F has two distinct local minima. It is however much more difficult to apply the variational method on $[a, b]$.

Remark 5. We note that obviously not all methods can be applied for a single equation. Let us consider $x^2 - c = 0$ over various segments $[a, b]$. When $c = 3$ and $a = 1$, $b = 2$ we note that only a1) applies. On the other hand when $a = -1$, $b = 1$ and $c = 0$ then we can employ a2). Methods described in b1) and b2) are in fact equivalent – this is however the case only for single variable mappings and that is why we do not pursue if further. In some cases method in b2) may overlap with c2) like for $f(x) = x^3$. Many likewise observations can be drawn when investigating the connection between all proposed methods.

Remark 6. After introducing these methods to the students, they saw that some would also apply to their example, where $f(x) = x^5 - 10x - 1$. Therefore, their projects would have been written quite differently have they known the course summary.

We finally mention that our methods, as depicted above and also these known from the high school mathematics, are also about the multiplicity. Indeed, when we consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad \Delta = b^2 - 4ac,\ a \neq 0$$

then we arrive at the existence and the uniqueness ($\Delta = 0$) and the multiplicity ($\Delta > 0$). As for methods mentioned as a1) and a2) in the above there are no means to argue about the uniqueness, while we have discussed about the multiplicity and properties of the sets of solutions for methods described as b1) and b2).

4. Final remark

There are some conclusions that are to be drawn from our observations in this note. These are about the fact that while teaching we should always have in mind future directions which are not restricted to subsequent notions, but rather to real world applications or at least apparent real world applications. We must be prepared to read known notions and facts about them from a new perspective as we did with rewriting the equation using the derivative, or else by defining the new attitude toward the meaning of word "solve". Thus the observation about equation (2), known as the Hadamard programme, leads to many important applications in engineering, numerical methods and related topics, see for example [8], and provides a good starting point in studying of more advanced topics of nonlinear analysis. We believe that it is possible to treat many other aspects of mathematical analysis and other course in such a way. The authors focused on the equation oriented approach due to the first authors research experience in nonlinear analysis connected to solvability of both potential and non-potential equations.

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M.G.: Institute of Mathematics, Lodz University of Technology, Al. Politechniki 8, 90-924 Lodz, Poland

E-mail: marek.galewski@p.lodz.pl

J.L.: Institute of Mathematics, Lodz University of Technology, Al. Politechniki 8, 90-924 Lodz, Poland

E-mail: jakub.lompies@gmail.com