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A GLIMPSE INTO THE THEORY OF CHESSBOARD COMPLEXES

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Dedicated to Professor Milosav M. Marjanović
on the occasion of his 90th birthday

Abstract. In this paper we review some of recent developments related to chessboard complexes and their generalizations. Focusing on explicit shelling constructions we have opportunity to present some surprising and exciting recent progress, to the necklace-splitting problem and to envy-free division.

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1. Introduction

The *chessboard complex* $\Delta_{m,n}$ is an abstract simplicial complex defined on an $m \times n$ chessboard with m columns and n rows (in the Cartesian notation or vice versa in the matrix notation). This complex appears in many incarnations (as a coset complex of the symmetric group, the matching complex in a complete bipartite graph, the complex of all injective functions, etc.). Its topological properties (high connectivity and the structure of an orientable pseudomanifold) have played a fundamental role in the proof of some deep results of topological combinatorics and discrete geometry (colored Tverberg theorems), see [21] for a survey.

Already in our first contact with chessboard complexes [22], almost thirty years ago, we had to prove that the complex $\Delta_{m,n}$ is $(m-1)$-dimensional, $(m-2)$-connected simplicial complex for $n \geq 2m-1$.

A canonical way to prove that a simplicial complex is highly connected is to show that it admits a *shelling*, a method which was not available at the time of writing [22].

In this paper we review some of the more recent developments related to chessboard complexes and their generalizations. Focusing on explicit shelling constructions (Section 5) we have opportunity to present some surprising and exciting recent progress, in the necklace-splitting problem (Section 3) and in the problem of envy-free division (Section 4).

1.1. The chessboard complex $\Delta_{m,n}$

The vertices of $\Delta_{m,n}$ are mn squares of the $m \times n$ chessboard and $(k-1)$-dimensional faces of $\Delta_{m,n}$ are all configurations of k non-taking (non-attacking) rooks, meaning that two rooks are not allowed to be in the same row or the same column.

We label the squares of the $m \times n$ table by (i, j), where i represents the column (numbered from left to the right) while j represents the row (numbered from top to the bottom).

2. Shellability of simplicial complexes

A d-dimensional simplicial complex K is called *pure* if its maximal simplices are all of the same dimension d. An ordering

$$F_1, F_2, F_3, \ldots, F_k, \ldots \tag{2.1}$$

of maximal simplices of a finite, pure d-dimensional simplicial complex K is called a *shelling* if the intersection

$$B_k = \Big(\bigcup_{i=1}^{k-1} F_i\Big) \cap F_k$$

is pure of dimension $d-1$ for all $k = 2, 3, \cdots$.

Exercise. An ordering (2.1) is a shelling if and only if for all $i < k$ there exist $j < k$ and $v \in F_k$ such that

$$F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{v\}\,.$$

In other words for each $k > 1$ and each $i < k$ the (possibly empty) intersection $F_i \cap F_k$ is contained in some "full intersection" $F_j \cap F_k$, where $j < k$ and $\dim(F_j \cap F_k) = d-1$.

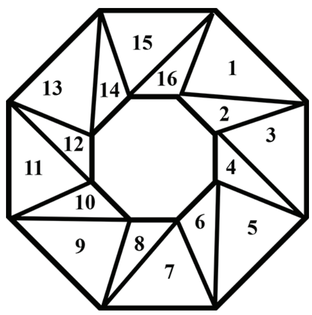


Fig. 1. A non-shelling of the annulus

Exercise. Why the ordering of triangles exhibited in Figure 1 is not a shelling?

Exercise. A graph without loops and multiple edges, interpreted as a 1-dimensional simplicial complex, is shellable if and only if it is connected.

Theorem 2.1. *A shellable d-dimensional complex is contractible or homotopy equivalent to a wedge of d-spheres*

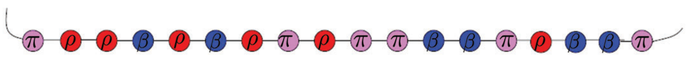
The case of graphs is elementary, however not without useful and amusing observations. For illustration let $\Gamma \subseteq K_{m,n}$ be a subgraph of the complete bipartite graph $K_{m,n}$. We know that Γ is shellable if it is connected. However, is it always possible to use a shelling on $K_{m,n}$ to obtain (by restriction) a shelling on the subgraph Γ? The following example shows that this is not the case.

Each edge $\{(i,1),(j,2)\}$ of $K_{m,n}$, for $i \in [m]$ and $j \in [n]$, is naturally associated a square $(i,j) \in [m]\times[n]$ in the $m\times n$ chessboard. A graph Γ (more precisely the set $E(\Gamma)$ of its edges) is naturally interpreted as a subset of the chessboard $[m]\times[n] = E(K_{m,n})$. Using this connection we easily observe the following interesting fact:

- The lexicographic shelling on $K_{n,n}$ is NOT a shelling on $\Delta_{n,2} \subset K_{n,n}$.

3. Fair division of a necklace

- There are r persons (agents, thieves) participating in a fair division of a necklace.
- There are n different types of gemstones (beads) in the necklace and the number of beads of each type is divisible by r.
- The thieves want to cut the necklace and fairly distribute the pieces so that each of them has equal number of beads of each type.
- The problem is to determine the *smallest number of cuts* which makes the fair distribution possible.



3.1. Fair division of a continuous necklace

- In a continuous version of the problem the necklace is the interval $[0,1]$.
- The distribution of gemstones is described by continuous measures μ_i $(i = 1,\dots,n)$.
- Interval $[0,1]$ is divided into the smallest possible number of pieces, which are collected into r groups $V_1,\dots,V_r$.
- The division is *fair* if all agents receive the same value of the necklace

$$\mu_i\Big(\bigcup V_j\Big) = \frac{1}{r}\mu_i([0,1]) \text{ for all } i = 1,\dots,n \text{ and } j = 1,\dots,r\,.$$

In Figure 2 we see a fair division of a continuous necklace, with 4 uniform measures supported by disjoint intervals, among 4 agents, with 12 cuts. Note that the number of pieces is $13 = 3\cdot 4+1$ and that the division is almost equicardinal in the sense that all agents receive as equal number of pieces as arithmetically possible.

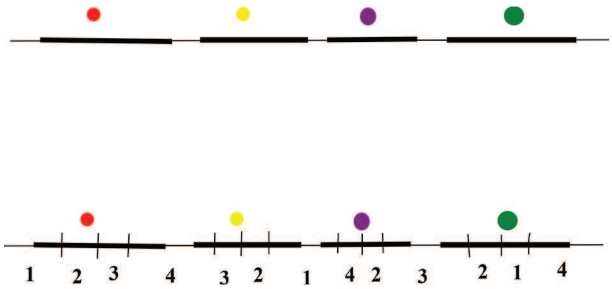


Fig. 2. A fair division of a continuous necklace with 4 thieves

Exercise. How many solutions are there? Is there a fair division which is not almost equicardinal? Is there a fair division where one of the thieves is given only two pieces of the necklace?

3.2. Theorem of Noga Alon

Theorem 3.1. (N. Alon [1]) *Let $\mu_1, \mu_2, \ldots, \mu_n$ be a collection of n (absolutely) continuous probability measures on $[0,1]$. Let $r \geq 2$ and $N := (r-1)n$. Then there exists a partition of $[0,1]$ by N cut points into $N+1$ intervals $I_1, I_2, \ldots, I_{N+1}$ and a function $f : [N+1] \to [r]$ such that for each μ_i and each $j \in [r]$,*

$$\sum_{p \in f^{-1}(j)} \mu_i(I_p) = 1/r\,.$$

The measures (mass distributions) exhibited in Figure 2 are quite special. The theorem of Alon says that, given r agents and n arbitrary mass distributions on $[0,1]$, one can always construct a fair division with the same number $(r-1)n$ of cuts, needed for a fair division of pairwise disjoint subintervals.

3.3. Almost equicardinal fair splitting

Example 3.2. Assume that the measures μ_j $(j = 1, \ldots, n)$ are supported by pairwise disjoint subintervals of $[0,1]$. In this case we need precisely $(r-1)n$ cuts which dissect the necklace into $(r-1)n+1$ parts. For this choice of measures if

$$\frac{(r-1)n+1}{r} = k + \frac{s}{r} \ (0 \leq s < r)$$

then there exists a fair partition/allocation of measures to r agents, which is almost equicardinal in the sense that and each agent is given either k or $k+1$ pieces of the necklace.

(A solution in the case $r = 4$ is exhibited in Figure 2.)

N. Alon proved his Theorem 3.1 in [1] and included this result in his lecture at the 1990 ICM in Kyoto, as one of the results which require non-constructive (topological) methods for their proof. It is interesting that the following natural question was asked more than thirty years later and answered positively in [10], for the case when r is a prime power .

Problem 3.3. For a given collection $\{\mu_j\}_{j=1}^n$ of absolutely continuous measures on $[0,1]$ and r agents, is it always possible to find a fair, almost equicardinal partition/allocation of the necklace.

Theorem 3.4. (D. Jojić, G. Panina, R.T. Živaljević [10]) *For given positive integers r and n, where $r = p^\nu$ is a power of a prime, let $k = k(r,n)$ and $s = s(r,n)$ be the unique non-negative integers such that $(r-1)n+1 = kr+s$ and $0 \leq s < r$. Then for any choice of n continuous, probability measures on $[0,1]$ there exists a fair partition/allocation of the associated necklace with $(r-1)n$ cuts which is also (k,s)-equicardinal in the sense that:*

- *each thief (agent) gets no more than $k+1$ parts (intervals);*
- *the number of thieves receiving exactly $k+1$ parts is not greater than s.*

Conjecture 3.5. Note that Theorem 3.4 is a generalization of Theorem 3.1 only if the number of agents is a prime power, $r = p^k$. We conjecture that this condition is necessary in the sense that if r is not a prime power Theorem 3.4 is no longer true.

3.4. Equicardinal fair splitting

Corollary 3.6. (Equicardinal necklace-splitting theorem) *In the special case $s = 0$, or equivalently if $(r-1)n+1$ is divisible by r, the Almost equicardinal necklace-splitting theorem guarantees the existence of a fair partition/allocation which is equicardinal in the sense that each thief is allocated exactly the same number of pieces of the necklace. Here we tacitly assume that the necklace is generic, i.e. that all $(r-1)n$ cuts are needed.*

3.5. Classical configuration space for the necklace splitting

- Each division of the necklace is described by a pair (x,f), a *partition/allocation* of the necklace.
- A partition x of the necklace $[0,1]$ is a sequence

$$0 = x_0 \leq x_1 \leq \ldots \leq x_N \leq x_{N+1} = 1\,.$$

- $f : [N+1] \to [r]$ is an allocation function.

The configuration space (generalized chessboard complex) which parameterizes all partitions/allocations (x,f) is the simplicial complex

$$(\Delta^N)^{*r}_{\Delta} \cong [r]^{*(N+1)} \cong \{(x_1-x_0)E_{1,f(1)} + (x_2-x_1)E_{2,f(2)} + \cdots + (x_{N+1}-x_N)E_{N+1,f(N+1)}\}_{(x,f)} \tag{3.1}$$

where $E_{i,j}$ is the $(r \times (N+1))$-matrix which has coefficient 1 at (i,j) and all other coefficients are 0.

Here it is not necessary to know the exact meaning of $(\Delta^N)^{*r}_{\Delta}$ and $[r]^{*(N+1)}$ (for the definitions of "joins" and "deleted joins" of complexes the reader is referred to [13]). Instead, one should recognize in the configuration space of all matrices (3.1), parameterized by all partitions/allocations (x, f), the geometric realization of the (relaxed) chessboard complex where the rooks are non-attacking only in the columns of the chessboard $[r] \times [N+1]$.

3.6. Naive configuration space for equicardinal splitting

- In (almost) equicardinal necklace-splitting theorem all thieves receive (almost) the same number of pieces of the necklace.
- The corresponding configuration space is the subspace $\mathcal{C}_1 \subset (\Delta^N)^{*r}_{\Delta}$ of all partitions/allocations (x, f) such that the sets $f^{-1}(i)\,(i \in [r])$ are all of (almost) the same cardinality.
- The naive configuration space $\mathcal{C}_1$ is N-dimensional, however it is not $(N-1)$-connected, the corresponding Borsuk-Ulam type theorem is not true and the "usual proof" of Theorem 3.1 (see [13]) breaks down!

3.7. Refined configuration space for equicardinal splitting

- (New idea) Initially allow a larger number of cuts $N = (r-1)(n+1)$ (before it was $(r-1)n$) and force some of these cuts to be superfluous by an appropriate choice of the configuration space.
- *Refined configuration space* $\mathcal{C}_2$ is defined as the simplicial complex of all rook placements on the $(N+1) \times r$ chessboard where $N = (r-1)(n+1)$ and:
  1. In each column there is at most one rook.
  2. In each row there are at most $(k+1)$ rooks.
  3. The number of rows where there are exactly $(k+1)$ rooks is at most s.
- Formally $\mathcal{C}_2$ is the symmetric multiple chessboard complex $\Sigma(\Delta^{\mathbf{k},\mathbf{1}}_{N+1,r}) = \Sigma(\Delta^{k_1,\dots,k_r;\mathbf{1}}_{N+1,r})$ with parameters $k_1 = \cdots = k_s = k+1$ and $k_{s+1} = \cdots = k_r = k$, see [12, Definition 2.1].
- Alternatively, the configuration space $\mathcal{C}_2$ can be described [12, Definition 2.3] as the symmetrized deleted join $SymmDelJoin(K_1, \dots, K_r)$ where

$$K_1 = \cdots = K_s = \binom{[N+1]}{\leqslant k+1}, K_{s+1} = \cdots = K_r = \binom{[N+1]}{\leqslant k}.$$

3.8. The configuration space $\mathcal{C}_2$ has desired properties

Theorem 3.7. (D. Jojić, S. Vrećica, R.Živaljević, [12])

- *Let $r, n \geq 2$ and let $rk + s = (r-1)n+1$ where k and s are the unique integers such that $k \geq 1$ and $0 \leq s < r$.*
- *Let $N = (r-1)(n+1)$ and $m = N+1$.*

Then the symmetric deleted join $SymmDelJoin(K_1, \dots, K_r)$ is $(m-r-1)$-connected where

$$K_1 = \cdots = K_s = \binom{[N+1]}{\leqslant k+1}, \quad K_{s+1} = \cdots = K_r = \binom{[N+1]}{\leqslant k}.$$

3.9. The original motivation for proving Theorem 3.7

Theorem 3.7 was proved by a shelling argument, similar but more complex then the argument presented in Section 5 for standard chessboard complexes. Its main application in [12] is the following theorem.

THEOREM 3.8. (Balanced generalized van Kampen-Flores theorem, [12, Theorem 1.2]) *Let* $r \geq 2$ *be a prime power,* $d \geq 1$*,* $N \geq (r-1)(d+2)$*, and* $rk+s \geq (r-1)d$ *for integers* $k \geq 0$ *and* $0 \leq s < r$*. Then for every continuous map* $f : \Delta^N \to \mathbb{R}^d$*, there are* r *pairwise disjoint faces* $\sigma_1, \dots, \sigma_r$ *of* Δ^N *such that* $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$*, with* $\dim \sigma_i \leq k+1$ *for* $1 \leq i \leq s$ *and* $\dim \sigma_i \leq k$ *for* $s < i \leq r$*.*

Theorem 3.8 confirmed a conjecture from [8] (Conjecture 6.6). It extends and unifies many earlier results including the following.

- Implies positive answer to the 'balanced case' of the problem whether each *admissible* r-tuple is *Tverberg prescribable*, [8, Question 6.9].
- The classical van Kampen-Flores theorem is obtained if d is even, $r = 2$, $s = 0$, and $k = \frac{d}{2}$.
- The sharpened van Kampen-Flores theorem [8, Theorem 6.8] corresponds to the case when d is odd, $r = 2$, $s = 1$, and $k = \lfloor \frac{d}{2} \rfloor$.
- The case $d = 3$ of the 'sharpened van Kampen-Flores theorem' is equivalent to the Conway-Gordon-Sachs theorem which says that the complete graph K_6 on 6 vertices is "intrinsically linked".
- The generalized van Kampen-Flores theorem [8, Theorem 6.3], which improves upon earlier results of Sarkaria and Volovikov, follows for $s = 0$ and $k = \lceil \frac{r-1}{r} d \rceil$.

4. Envy-free division of a cake

Given a commodity (resource) and a set of agents (players), one of the goals of *welfare economics* (`https://en.wikipedia.org/wiki/Welfare_economics`) is to divide the resource among the agents in an *envy-free* manner. Envy-freeness is the principle where every player feels that their share is at least as good as the share of any other agent, and thus no player feels envy. Here is an example.

- A birthday party is planned for r children.
- Children may have different taste and preferences.
- The birthday cake is divided into r pieces $\{V_i\}_{i=1}^r$
$$Cake = V_1 \sqcup V_2 \dots \sqcup V_r\,.$$
- A division of the cake is *envy-free* if each child is satisfied with their piece of cake and doesn't want to trade with any other child.
- In other words there is a permutation $\pi : [r] \to [r]$ which connects each child (labeled by $i \in [r]$) with the piece $V_{\pi(i)}$ she prefers.

The classical approach to envy-free division and equilibrium problems arising in mathematical economics typically relies on Knaster-Kuratowski-Mazurkiewicz theorem, Sperner's lemma or some extension involving mapping degree. Here we outline a different and relatively novel approach, originally developed in [10] and [15], where the emphasis is on configuration spaces and equivariant topology with chessboard complex $\Delta_{r,2r-1}$, as the main character.

We illustrate the method by proving an extension of the classical envy-free division theorem of Stromquist, Woodall, and Gale, where the emphasis is on preferences allowing the players to choose degenerate pieces of the cake, see [4, 17, 19].

4.1. The difference between envy-free and fair division

- In fair division there is an objective (external) criterion of fairness. All players agree about the value of the pieces and have the same preferences.
- In envy-free division each player has her own preferences which may be unknown or even irrational (from the view point of other players).
- In fair division an arbitrary permutation of pieces is still a fair division.
- In envy-free division, if it exists at all, the pieces usually cannot be exchanged.

4.2. A mathematical model of the cake division

There are r players who want to divide among themselves a commodity, referred to as the cake. In a mathematical simplification a basic model of the cake is the interval $I = [0, 1]$, which should be cut into r pieces by $r - 1$ cuts. Therefore a partition (cut) of the cake (in this model) is a sequence of numbers $x = (x_1, \dots, x_{r-1})$ where

$$0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{r-1} \leqslant 1\,. \tag{4.1}$$

The pieces of the cake arising from this division are the closed intervals (tiles)

$$I_i = I_i(x) := [x_{i-1}, x_i]\,(i = 1, \dots, r), \text{ where } x_0 = 0 \text{ and } x_r = 1$$

while $\lambda_i = x_i - x_{i-1}$ are the corresponding barycentric coordinates.

- The configuration space, parameterizing all possible cuts, is the standard $(r-1)$-dimensional simplex in $\mathbb{R}^r$

$$\Delta^{r-1} = \{\, \lambda_1 e_1 + \cdots + \lambda_r e_r \mid \lambda_1 + \cdots + \lambda_r = 1 \text{ and } (\forall k)\, \lambda_k \geq 0 \,\}\,.$$

Given a cut $x \in \Delta^{r-1}$, each player should make up her mind and choose one or more pieces that they like the most or prefer to the rest of the pieces by some subjective preference. What is the mathematical equivalent of this procedure?

There are several possibilities how to express preferences in a mathematical language. Here we use one of the simplest models which nevertheless preserves most of the features of more general constructions.

- Each player (labelled by $j \in [r]$) has her own measure ν_j she uses for evaluating the pieces of the cake. A tile I_k is preferred if and only if

$$\nu_j(I_k) \geq \nu_j(I_i) \text{ for each } i \in [r]$$

  or in other words if $\nu_j(I_k) = M_j = M_j(x) := \max\{\nu_j(I_i)\}_{i=1}^r$.

In this model all measures ν_j are continuous (with respect to the Lebesgue measure μ)

$$\nu_j(A) = \int_A \phi_j d\mu$$

where $\phi_j : [0,1] \to \mathbb{R}$ is a measurable density function and $A \subseteq [0,1]$ a measurable set.

The densities ϕ_j are not necessarily positive, so in general we deal with *signed measures*, meaning that if $\nu_j(J) < 0$ then the player j would prefer the empty set to J. This is quite reasonable and happens for example in a rent-partition or chore-division problem where the divided resource is undesirable, so that each participant wants to get as little as possible.

On the other hand if the players divide something desirable, such as a cake or a French loaf (Baguette) then $\phi_j \geq 0$ is a reasonable assumption and in that case we talk about "hungry players". Note that the loaf of bread can be partially burnt so the corresponding density function may be both positive and negative.

4.3. Envy-free division with hungry players

The following theorem, associated with the names Stromquist, Woodall, and Gale [9, 17, 19], claims that an envy-free division is always possible if the players are hungry.

THEOREM 4.1. *Suppose that the preferences of all players are described by positive measures ν_j $(j = 1, \dots, r)$, with associated non-negative density functions $\phi_j \geq 0$. Then there exists a cut $x \in \Delta^{r-1}$ and a permutation $\pi \in S_r$ such that for the corresponding partition $\{I_k(x)\}_{k=1}^r$ of the interval* $[0,1]$

$$\nu_j(I_{\pi(j)}(x)) \geq M_j(x) := \max\{\nu_j(I_k(x)) \mid k = 1, \dots, r\}$$

for each $j = 1, \dots, r$.

Proof. Choose $\epsilon > 0$. For each player $j \in [r]$ define $A_i^j \subseteq \Delta^{r-1}$ as the set of all cuts $x \in \Delta^{r-1}$ such that $\nu_j(I_i(x)) \geq M_j(x)$. In other words A_i^j collects all cuts where the player j prefers the tile I_i. By continuity of the measure A_i^j is a closed set. Moreover $\{A_i^j\}_{i=1}^r$ is a covering of Δ^{r-1} for each j, since in each cut a player prefers some of the tiles.

Similarly, let $O_i^j = \{x \in \Delta^{r-1} \mid \nu_j(I_i(x)) > M_j(x) - \epsilon\}$ be an open superset of A_i^j. Let $g_i^j : \Delta^{r-1} \to [0,1]$ be a continuous function such that

$$g_i^j(A_i^j) = \{1\} \quad \text{and} \quad g_i^j(\Delta^{r-1} \setminus O_i^j) = \{0\}\,.$$

By construction the functions

$$f_i^j = \frac{g_i^j}{g^j} \qquad \text{where} \qquad g^j := g_1^j + \cdots + g_r^j$$

create a partition of unity, in the sense that $f_1^j(x) + \cdots + f_r^j(x) = 1$ for each $x \in \Delta^{r-1}$. Moreover, if $f_i^j(x) > 0$ then $x \in O_i^j$.

Note that the condition that the players are hungry (and never choose degenerate tiles) translates into the condition that for all i and j

$$f_i^j(\Delta_i^{r-1}) = \{0\} \quad \text{where} \quad \Delta_i^{r-1} := \{x \in \Delta^{r-1} \mid \lambda_i = 0\}\,. \tag{4.2}$$

Note that the map $h_i = \frac{1}{r}(f_i^1 + f_i^2 + \cdots + f_i^r)$ also satisfies the condition (4.2). It follows that $h_i(\Delta_i^{r-1}) \subseteq \Delta_i^{r-1}$ for all i which implies that the restriction of the map

$$h = (h_1, h_2, \ldots, h_r) : \Delta^{r-1} \longrightarrow \Delta^{r-1} \tag{4.3}$$

to the boundary $\partial\Delta^{r-1}$ is homotopic (by the linear homotopy) to the identity map.

It follows that the degree of this map is equal to one and, as a consequence, $h(x) = (\frac{1}{r}, \ldots, \frac{1}{r})$ for some $x \in \Delta^{r-1}$.

Summarizing, we have established the existence of a cut x such that the matrix $M = (f_i^j(x))$ is a bistochastic matrix. By Birkhoff-von Neumann theorem M is a convex combination of permutation matrices. Each of these matrices produces a permutation $\pi \in S_r$ such that $f^j_{\pi(j)}(x) > 0$ (and consequently $x \in O^j_{\pi(j)}$) for each $j \in [r]$.

Finally, by choosing a zero sequence $\epsilon_n \to 0$ we obtain a sequence $x_n \in O^{j,\epsilon_n}_{\pi(j)}$, which can be assumed to be convergent, $x_n \to x$. Since

$$\bigcap_{n\in\mathbb{N}} O^{j,\epsilon_n}_{\pi(j)} = A^j_{\pi(j)}$$

we conclude that $x \in A^j_{\pi(j)}$ for each $j \in [r]$, which completes the proof of the theorem. ■

4.4. Envy-free division with not necessarily hungry players

Condition (4.2) played a decisive role in the proof of Theorem 4.1. What can be said in the absence of this condition?

Avvakumov and Karasev [4], extending some partial results of Segal-Halevi [16] and of Meunier and Zerbib [14], almost forty years after the appearance of [17] and [19], proved a general result where (under some conditions) the players are allowed to prefer and choose an “empty piece” (= a degenerate tile), if all other tiles are less satisfactory. The proof they offered is also based on a degree theoretic argument and relies on the simplex, as the configuration space, as the proof of Theorem 4.1. However their proof is more complex and more technical.

Here we present a simpler and possibly more conceptual proof of the result of Avvakumov and Karasev, which uses equivariant topology and chessboard complexes as configuration spaces. This proof was originally discovered in [10], as part of the proof of (more general) result [10, Corollary 6.10]. Another presentation of this proof, more succinct and with all details included can be found in [15].

As in the case of Theorem 4.1, by sacrificing some generality, we formulate and prove a version involving the preferences defined in terms of signed measures.

Theorem 4.2. *Suppose that the preferences of all players are described by signed measures ν_j $(j = 1, \ldots, r)$, with associated density functions ϕ_j. Then there* *exists a cut* $x \in \Delta^{r-1}$ *and a permutation* $\pi \in S_r$ *such that for the corresponding partition* $\{I_k(x)\}_{k=1}^r$ *of the interval* $[0,1]$

$$\nu_j(I_{\pi(j)}(x)) \geq M_j(x) := \max\{\nu_j(I_k(x)) \mid k = 1, \dots, r\}$$

for each $j = 1, \dots, r$.

Proof. (outline) We mimic the proof of Theorem 4.1 with some important modifications. The most important difference is the use of the chessboard complex $\Delta_{r,2r-1}$, instead of the simplex Δ^{r-1}, as the main configuration space.

In particular the map (4.3) will be replaced by an equivariant map $h : \Delta_{r,2r-1} \to \Delta^{r-1}$. The shellability of the complex $\Delta_{r,2r-1}$, established in Section 5, implies that $\Delta_{r,2r-1}$ is a $(r-2)$-connected, $(r-1)$-dimensional simplicial complex with a free action of the group $G = (\mathbb{Z}_p)^k$ (here we use the fact that $r = p^k$ is a prime power).

The sets A_i^j and O_i^j are now defined as (G-invariant) subspaces of the chessboard complex $\Delta_{r,2r-1}$. For example $(x, f) \in A_i^j$ means that

$$\nu_j(I_{f^{-1}(i)}(x)) \geq M_j(x) := \max\{\nu_j(I_k(x)) \mid k = 1, \dots, r\}\,.$$

Let (f_i^j) be a matrix of functional preferences associated to the matrix of preferences (A_i^j), subordinated to (O_i^j), which is also equivariant in the sense that

$$f^j_{\sigma(i)}(\sigma(x,\alpha)) = f_i^j(x,\alpha)\,. \tag{4.4}$$

In particular (f_i^j) is a collection of functions $f_i^j : \Delta_{r,2r-1} \to [0,1]$ satisfying the following conditions.

1. For each $j \in [r-1]$ the collection $\{f_i^j\}_{i=1}^r$ is a partition of unity
$$f_1^j + f_2^j + \dots + f_r^j = 1\,.$$
2. For each i and j
$$A_i^j \subseteq \{x \mid f_i^j(x) > 0\} \subseteq O_i^j\,.$$

Let $h_i = \frac{1}{r}(f_i^1 + f_i^2 + \dots + f_i^r)$. The map

$$h = (h_1, h_2, \dots, h_r) : \Delta_{r,2r-1} \longrightarrow \Delta^{r-1}$$

is equivariant and, since $\Delta_{r,2r-1}$ is $(r-2)$-connected, Volovikov's theorem [13] guarantees that $h(x,\alpha) = (\frac{1}{r}, \dots, \frac{1}{r})$ for some $(x,\alpha) \in \Delta_{r,2r-1}$.

As in the proof of Theorem 4.1, we established the existence of a partition/allocation (x, f) such that the matrix $M = (f_i^j(x,f))$ is a bistochastic matrix. After that the proof is completed by a similar argument as the proof of Theorem 4.1. ∎

5. Shellability of the chessboard complex $\Delta_{m,n}$

G. Ziegler proved in [20] that chessboard complexes $\Delta_{m,n}$ are vertex decomposable for $m \geqslant 2n-1$. As vertex-decomposability is a stronger property than shellability (see [6]), it follows that chessboard complexes are shellable. We give a different proof of this result, illustrating the central ideas, which can be with some modifications applied (as it was done in [11] and [12]) to other classes of (generalized) chessboard complexes.

Perhaps the main initial obstacle, as remarked by Ziegler in [20], is that the natural lexicographical order of facets of $\Delta_{n,m}$ is not a shelling order.

Note that the whole chessboard complex $\Delta_{m,n}$ can be obtained as a union of m cones

$$\Delta_{m,n} = \bigcup_{i=1}^{m} (i,1) * K_i, \tag{5.1}$$

where $K_i \cong \Delta_{m-1,n-1}$ is the chessboard complex on the board obtained by deleting the first row and i-th column from the chessboard $[m] \times [n]$. If we knew, by the inductive assumption, that $\Delta_{m-1,n-1}$ is shellable, it would be natural to interpret this shelling as a shelling of $(i,1) * K_i$, with the idea to extend it to the whole complex $\Delta_{m,n}$.

However this plan does not work in general since, as remarked by Ziegler, there are difficulties if we use the natural order $1 < \cdots < i-1 < i+1 < \cdots < n$ of the columns of K_i (see also the remark at the end of Section 2.

We will show how to overcome this difficulty, by using the symmetry of chessboard complexes, and utilizing the following linear order $<_i$ of the columns of K_i:

$$i+1 <_i i+2 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1. \tag{5.2}$$

REMARK 5.1. A discrete set is shellable with respect to any order. We start by fixing $1 < 2 < \ldots < m$ as the shelling order of $\Delta_{m,1}$.

We describe a shelling order on $\Delta_{m,n}$ recursively, by assuming that a shelling order on complexes $\Delta_{k,r}$ is already defined, for all $r < n$ and $k \geqslant 2r-1$. The facets of $\Delta_{m,n}$ are ordered by the following guiding principles.

(1) The position of the rook in the first row.

Recall that each facet of $\Delta_{m,n}$ contains exactly one rook in the first row. Our shelling order starts with the facets of $\Delta_{m,n}$ having a rook at the position $(1,1)$, followed by the facets with a rook at the position $(2,1)$, etc. The shelling order finishes with the facets that contain a rook at the position $(m,1)$. In other words, all facets of $(i,1) * K_i$ come in front of the facets of $(j,1) * K_j$ for $1 \leqslant i < j \leqslant m$. Here we use the decomposition of $\Delta_{m,n}$ defined in (5.1).

Now, we describe the order of facets in each $(i,1) * K_i$. To order the facets of $\Delta_{m,n}$ that have rook at $(i,1)$ for $i > 1$ we consider:

(2) The number of occupied columns immediately before the i-th column.

The shelling order of the facets containing the rook at $(i,1)$ starts with facets that do not contain a rook in the column $(i-1)$. These facets span a subcomplex of $\Delta_{m,n}$ that is isomorphic to $\Delta_{m-2,n-1}$ (we delete the first row and columns i and $i-1$). By the inductive assumption this subcomplex is shellable. We order this class of facets of $\Delta_{n,m}$, following the assumed shelling order on $\Delta_{n-1,m-2}$.

The order of the facets of $\Delta_{n,m}$ that contain a rook at the position $(i,1)$ continues with the facets that have a rook in the column $i-1$ but not in the column $i-2$. Note that the subcomplex of $\Delta_{m,n}$ spanned by the facets that contain the rooks at $(i,1)$ and $(i-1,j)$ (for a fixed $j>1$), but do not contain a rook in the column $i-2$ is isomorphic to $\Delta_{m-3,n-2}$ (here we delete two rows and three columns). Again, we use the assumption of shellability of $\Delta_{m-3,n-2}$ to define the order of corresponding facets of $\Delta_{m,n}$. All such facets that contain a rook at $(i-1,p)$ precede the facets that contain the rook at $(i-1,q)$ if $p<q$.

Our shelling order of the facets containing a rook at $(1,i)$ continues further in the same manner. We order the facets that have a rook at $(i,1)$, contain the rooks in the columns $i-1,\dots,i-k+1$ (at fixed positions), but not in the column $i-k$. Now, we delete column i, the last k columns $i,i-1,\dots,1,m,\dots,m-k+i$ in the order defined in (5.2) and first k rows. The remaining part of the table spans a subcomplex of $\Delta_{m,n}$ isomorphic to $\Delta_{m-k-1,n-k}$, which is again shellable by assumption. For a fixed configuration of the rooks in the columns $i-1,\dots,i-k+1$ (there are $(n-1)(n-2)\cdots(n-k+1)$ such configurations) the shelling order for $\Delta_{n-k,m-k-1}$ defines the order of corresponding facets of $\Delta_{n,m}$.

We illustrate the construction described above by some examples.

Example 5.2. We start with the shelling of $\Delta_{m,2}$ ($m\geqslant 3$) described above. Assume that the squares in the first row are $a_1,a_1,\dots,a_m$ and the squares in the second row are $b_1,b_2,\dots,b_m$. The chosen shelling order for $\Delta_{m,2}$ is

$$a_1b_2,a_1b_3,\dots,a_1b_m;a_2b_3,a_2b_4,\dots,a_2b_m,a_2b_1;\dots$$
$$\dots;a_ib_{i+1}\dots,a_ib_m,a_ib_1,\dots,a_{i-1};\dots,a_mb_1,\dots,a_mb_{m-1}.$$

After that we list the facets of $\Delta_{5,3}$ using the labels of the squares as shown in the following table

b_1	b_2	b_3	b_4	b_5
a_1	a_2	a_3	a_4	a_5
1	2	3	4	5

We apply the fixed shelling order for $\Delta_{4,2}$ described above, but we change the labels of columns. The shelling of $(1,1)*K_1$ is

	b_2	b_3	b_4	b_5
	a_2	a_3	a_4	a_5
1				

$1a_2b_3$ $1a_2b_4$ $1a_2b_5$ $1a_3b_4$ $1a_3b_5$ $1a_3b_2$
$1a_4b_5$ $1a_4b_2$ $1a_4b_3$ $1a_5b_2$ $1a_5b_3$ $1a_5b_4$

For $(2,1)*K_2$ we begin with the facets that do not contain a rook in the first column, and order them as in the shelling of $\Delta_{3,2}$:

		b_3	b_4	b_5
		a_3	a_4	a_5
	2			

$2a_3b_4$ $2a_3b_5$ $2a_4b_5$ $2a_4b_3$ $2a_5b_3$ $2a_5b_4$

Now, we list facets from $(2,1) * K_2$ that contain a rook in the first row, but not in the column 5.

		b_3	b_4	
a_1				
	2			

$2a_1b_3 \quad 2a_1b_4$

b_1				
		a_3	a_4	
	2			

$2b_1a_3 \quad 2b_1a_4$

Facets $2a_1b_5$ and $2b_1a_5$ from $(2,1) * K_2$ that contain rooks in the column 1 and the column 5 finish this part of the shelling of $\Delta_{5,3}$.

We continue by describing the shelling of the facets from $(3,1) * K_3$. First, we list facets with no rooks in the column 2:

b_1			b_4	b_5
a_1			a_4	a_5
		3		

$3a_4b_5 \quad 3a_4b_1 \quad 3a_5b_1 \quad 3a_5b_4 \quad 3a_1b_4 \quad 3a_1b_5$

These facets are followed by the facets containing a rook in the second column, but not in the first:

			b_4	b_5
	a_2			
		3		

$3a_2b_4 \quad 3a_2b_5$

	b_2			
			a_4	a_5
		3		

$3b_2a_4 \quad 3b_2a_5$

The last two facets at this stage are $3a_2b_1$ and $3b_2a_1$.

Now, we list all facets with the rook at the position $(4,1)$. We begin with the facets without a rook in the column 3:

b_1	b_2			b_5
a_1	a_2			a_5
			4	

$4a_5b_1 \quad 3a_5b_2 \quad 4a_1b_2 \quad 4a_1b_5 \quad 4a_2b_5 \quad 3a_2b_1$

followed by facets

b_1				b_5
		a_3		
			4	

$4a_3b_5 \quad 4a_3b_1$

		b_3		
a_1				a_5
			4	

$4b_3a_5 \quad 4b_3a_1$

The last two facets at this stage are $4a_3b_2$ and $4b_3a_1$.

Our shelling is completed by ordering of facets with the rook at the position $(5,1)$

b_1	b_2	b_3		
a_1	a_2	a_3		
				5

$5a_1b_2 \quad 5a_1b_3 \quad 5a_2b_3 \quad 5a_2b_1 \quad 5a_3b_1 \quad 5a_3b_2$

After these facets we list facets that have a rook in the column 4, but not in the column 3:

b_1	b_2			
			a_4	
				5

$5a_4b_1 \quad 5a_4b_2$

			b_4	
a_1	a_2			
				5

$5b_4a_1 \quad 5b_4a_2$

The last two facets are $5a_4b_3$ and $5b_4a_3$.

In our construction, the shellability of $\Delta_{m,n}$ follows from the shellability of the complex $\Delta_{m-2,n-1}$. This explains the condition $m \geqslant 2n-1$ in Ziegler's result.

Let us recapitulate the construction above by describing how to compare the facets of $\Delta_{m,n}$. For a facet $A = \{(1, a_1), \dots, (n, a_n)\}$ of $\Delta_{m,n}$ we define

(a) $i(A) = a_1$ is the position of the rook in the first row;

(b) $r(A) = r$ is the maximal $r \in \mathbb{N}$ such that A has rooks in the column a_1 and $r-1$ last consecutive columns in the order defined in (5.2). If $r < a_1$, then the column $a_1 - r$ is empty, and if $r \geqslant a_1$ the column $n - r + a_1$ is empty.

(c) Let $R(A) = (1, i_2, \dots, i_r)$ be the r-tuple of indices that encode the rows containing the rooks in r consecutive columns. If $r < a_1$ we have that $\{(a_1, 1), (a_1 - 1, i_2), \dots, (a_r - r + 1, i_r)\} \subseteq A$. We define $R(A)$ similarly if $r \geqslant a_1$.

Recall the definition of the lexicographical order of r-tuples

$$(a_1, a_2, \dots, a_r) <_L (b_1, b_2, \dots, b_r) \Leftrightarrow a_1 < b_1 \text{ or } a_i < b_i \text{ and } a_j = b_j \text{ for all } j < i.$$

We define the linear order $\ll$ on the facets of $\Delta_{m,n}$ as follows

$$A \ll B \Leftrightarrow \begin{cases} i(A) < i(B), & ; \text{or} \\ i(A) = i(B) \text{ and } r(A) < r(B), & ; \text{or} \\ i(A) = i(B), r(A) = r(B) \text{ and } R(A) <_L R(B), & ; \text{or} \\ R(A) = R(B) \text{ and } A' \ll B'. \end{cases} \tag{5.3}$$

Here A' and B' denote faces of A and B obtained by deleting their common rooks contained in the rows labelled by elements from $R(A) = R(B)$. Here we use the assumption that $\Delta_{m-r-1,n-r}$ is shellable.

The proof that $\ll$ is a shelling order is a special case of Theorem 4.4 in [12]. We will repeat shortly the arguments for the sake of completeness.

Case 1*:* $i(A) = a_1 < i(B) = b_1$.

If there exists an empty column $j < b_1$ in B we let $C = B - \{(b_1, 1)\} \cup \{(j, 1)\}$. If all columns from 1 to b_1 are occupied in B, let i denote the row that contains the rook in a_1-th column. In that case we let $C = B - \{(a_1, i)\} \cup \{(j, i)\}$, for some $j > b_1$.

Case 2*:* $i(A) = i(B) = a_1$, $r(A) = r_A < r_B = r(B)$.

In this case, the column $a_1 - r_A$ is empty in A, and if $(a_1 - r_A, i) \in B$, we let $C = B - \{(a_1 - r_A, i)\} \cup \{(j, i)\}$ for some empty column in B.

Case 3*:* $i(A) = i(B) = a_1$, $r(A) = r(B) = r$, $R(A) <_L R(B)$.

Let i_j be the first entry in $R(A) = (1, i_2, \dots, i_r)$ where R_A is lexicographically smaller than R_B. When we go from the a_1-th column to the left, this is the first column where the rook in A is below the rook in B. Let x and y denote the rows containing the rooks in the i_j-th column in A and B respectively. In that case we define $C = B - \{(i_j, y)\} \cup \{(p, y)\}$ for an adequate p.

Note that in any of the above cases we obtain $C \ll B$ such that C and B differ in just one vertex.

Case 4*:* If $R(A) = R(B)$ we compare A' and B', the facets of $\Delta_{m-r-1,n-r}$ (we delete $r+1$ columns a_1, $a_1 - 1, \dots, a_1 - r$ and r rows from $R(A)$). At this place we need the assumption about shellability of the chessboard complex on the smaller table. As $n \geq 2m - 1$ implies $n - r - 1 \geq 2(m - r)$ for all r, this condition is precisely what is needed.

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