THE TEACHING OF MATHEMATICS

2020, Vol. XXIII, 2, pp. 109–116

A CASE STUDY OF A STUDENT WHO CREATED PROBLEMS FOR A MATHEMATICS COMPETITION

Andreas Poulos

Abstract. In this article we present the conclusions drawn from our research on the subject of problem posing for mathematics competitions. We present the case study of a student who is familiar with problem solving, has understood what problem posing means and what is a mathematical problematic situation, has proven his mathematical skills, and is asked to create his own mathematical problem. The conclusions drawn from this research are, firstly, on the relation of mathematical creativity and mathematical problem posing, secondly, on the relation of mathematical education and self-education, and thirdly on the relationship between the creator of a problem and the potential solver of the problem.

MathEduc Subject Classification: D54

MSC Subject Classification: 97D50

Key words and phrases: problem solving; problem posing; mathematical creativity; problem situation.

The theoretical context of the research

The theoretical context of this research has three components.

The first component is "The research on how experts pose problems may also be profitable and lead to new ideas about how to improve problem-posing competences of school children and mathematics teachers" [4, p. 82].

The second component is "Thus, encouraging students to generate problems is not only likely to foster student understanding of problem situations, but also to nurture the development of more advanced problem-solving strategies" [10, p. 153].

The third component is "Less attention has been given to comments made by those who create problems, particularly when these are students. We need to know more about how problem posers think and react as they go about the task of posing problems" [10, p. 159].

It is known in the Didactics of Mathematics, and specifically, in the topic of problem posing, that the concept "mathematical situation" has multiple meanings, and there are differences between the various interpretations of this concept. In our research, we set the meaning of this concept to be: "A mathematical situation is not yet a problem. It consists of a set of mathematical objects, linked by some certain relations. With this basis, the participants must investigate the properties of the proposed situation, adding if necessary other elements, and to create one or more problems" [11, p. 31].

In our research, the mathematical situation was simple, in order to allow the young math problem creator many degrees of freedom. The only given data was a square. Given a square, the student (case study) was asked to create his own problem. Nothing further was given to the student.

One of the educational goals related to our research is the following goal described in [10]: "Are we giving those who pose problems time and space to reflect on problem posing, and are we as mathematics education researchers, taking time to listen to and learn from student voices?" [10, p. 160]. The article by J. Kilpatrick is written in a similar spirit: "Problem formulating should be viewed not only as a goal of instruction but also as a means of instruction. The experience of discovering and creating one's own mathematics problems ought to be part of every student's education. Instead, it is an experience that few students have today – perhaps only if they are candidates for advanced degrees in mathematics" [2, p. 123].

A second educational goal of our research is related to the question – challenge set by Robinson, regarding the education of gifted students: "Furthermore, to a large extent, when differentiation does take place, it frequently consists of assigning gifted children independent work that takes them away from teacher guidance and support. While this may constitute a better solution than none, we expect gifted children to become autodidacts at a very early age!" [9, p. 262].

We also take into careful consideration the research [1], who studied "How do two college students formulate and solve their own mathematical problems".

This paper is a continuation on research conducted in [7] and [8].

Research description

This research is a case study of a student, 15 years old, who for two years has systematically worked on problem solving contests, organized by the Hellenic Mathematical Society. The student has distinguished himself at the National Mathematical Olympiad of Greece. The researcher has met several times with the student and has explained what problem posing means and what is a mathematical problematic situation. The way the student understood the problem posing process is that this process relates to the modification and/or generalization of a problem, removing data, formulating new questions on the same problem, etc.

We asked the student to keep a detailed log of his efforts in creating a problem for a mathematics competition, where he would explain the train of thought, concerns and obstacles encountered in the formulation of the problem.

We present the student's log.

The teacher asked me to create a difficult problem based on the properties of a square. I thought of combining two problems I recently solved, while training for mathematical competitions. These two problems have similarities and differences.

In the first problem – call it No. 1 (see Fig. 1) – we take any point P inside the square $ABCD$. We are given the distances of the point P from three of the four corners of the square, and we are asked to calculate the side of the square.

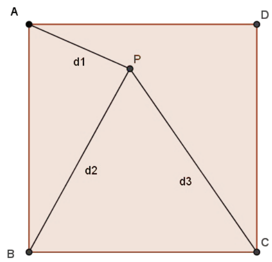

Fig. 1

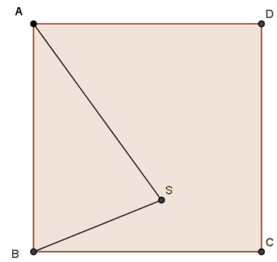

Fig. 2

In the second problem – call it No. 2 (see Fig. 2) – again we take any point S inside the square $ABCD$. We now know the length of the side of the square and the distances of the point S from two of the four corners of the square. The question is to find the distance of the point S from the other two corners of the square.

I started creating my own problem incorporating first problem No. 1, i.e., taking any point P inside the square and knowing the distances of the point P from three corners of the square. Practically speaking, I knew the length of the side of the square, but the person who was going to solve the problem wouldn't know it. Next, I took a second point S inside the square, and I took as a given the distances of S from two of the four corners of the square. According to problem No. 2, I could calculate the distances of the point S from the other two corners of the square.

The shape I drew (see Fig. 3) gave me the idea of creating a problem with the triangle with corners P, S and A (A being the corner of the square closest to P). At first, I considered a difficult problem calculating the area of the triangle PSA; I quickly realized this was not as difficult as I expected.

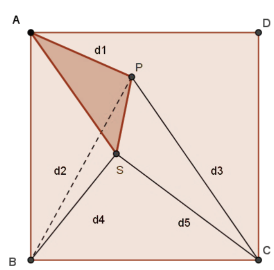

Fig. 3

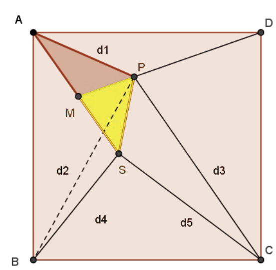

Fig. 4

I then saw that in the shape (see Fig. 4) there was a smaller triangle PSM, including again the side PS, and with as third corner the point M of the intersection of the line segments PD and AS.

At first I could not imagine how to calculate the area of the triangle SPM. I then thought of Heron's formula to calculate the area of a triangle, when we know the lengths of its three sides. Therefore, to calculate the area of the triangle SPM, it was enough to calculate the lengths of the sides PM and SM. I assume that I can calculate the length of PS, which is the "key" to solving the problem I will create.

Now, I state the final form of the problem I created.

> *Inside a square $ABCD$ there are two distinct points P and S. The point P is closer to the corner A and the point S is closer to the corner B. The distances between point P and the three corners A, B, C, are denoted respectively d_1, d_2, d_3. The distances between point S and the corners B and C are respectively d_4 and d_5. The line DP intersects the line AS at point M. Calculate the area of the triangle PMS in terms of the lengths d_1, d_2, d_3, d_4, d_5.*

I had to ensure that the problem I created has a solution. I first thought of using Analytic Geometry, even though I do not particularly like its methods because most times they require a lot of algebraic operations. Another reason I chose Analytic Geometry is that I could not see at first a solution to the problem using Euclidean Geometry.

To calculate PS I use the formula

$$PS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \tag{1}$$

where x_1, y_1 and x_2, y_2 are the coordinates of the points P and S, respectively. These coordinates are known, because we know the distances of the points P and S from the corners of the square $ABCD$.

Using the same formula (1) I need to find the distances PM and SM. In other words, I need to find the coordinates of the point M. This point is the point of intersection of the two lines AS and DP. Since the coordinates of the four points A, S, D, and P are known, we can calculate the coordinates of the point M.

Continuing, to solve the problem, I had to assign specific numbers to the distances d_1, d_2, d_3, d_4, d_5. Assigning these numbers, I ended up with a polynomial equation of 2^{nd} degree. This equation had negative discriminant. Then I realized that the values for the distances d_i, $i = 1, 2, 3, 4, 5$ cannot be arbitrary, because the problem I created might not have a solution. Therefore, I experimented a little with the values of d_i to have "good" results. I also had to take into consideration the triangular inequality $b - c < a < b + c$ for the lengths $a > b > c$ of the sides of the triangle.

However, another unexpected finding came up. The side of the square has two values (which is expected, since it is the root of a quadratic equation), an integer value and an irrational value. Therefore, I thought of adding to the data of the problem the extra restriction that "we know that the side of the square is an integer number", something I had not written in the initial statement of the problem.

I think the problem that I created is not very difficult, because the necessary theorems to solve this problem is the Pythagorean Theorem, polynomial equations of 2^{nd} degree, and Heron's formula for the calculation of the area of a triangle. Nonetheless, it is a problem appropriate for a mathematics competition, since it asks of the solver to combine various theorems and make many efforts to solve it.

I wish I could solve my problem using Euclidean Geometry.

A.V. K.

We continue with an interview in the form of questions towards the student. These questions had as a goal to clarify a few points of the student's text, so that his line of thought in creating this problem could be understood as much as possible.

Question 1. Where did you find the problem with the known distances of a point from the vertices of a square?

Answer. We had to solve this problem when we were preparing for mathematical competitions. For a moment, I thought about changing the known distances of the point from the vertices of the square, to known distances of the point from the vertices of a rectangle, but then I decided to keep it as I initially encountered it. Besides, one needs to try a lot to solve this problem, and I was asked to create a problem with a square.

Question 2. Where did you find the second problem with the known distances of a point from the vertices of a square?

Answer. This problem is my creation. Using the first problem as a foundation, I thought of a modification. I think these two problems are equivalent, that is, they both use the same techniques in their solutions.

Question 3. How did you think to create the final problem?

Answer. I think the idea came to me very quickly. From the moment I chose to use Problem 1 as a starting point, I immediately thought of a change in the data, in making the side of the square known, and making one of the distances of the point from one of the vertices of the square unknown. I then used two different points to hide – as best as I could – the two initial problems I used.

Question 4. Why didn't you add as a problem the area of the triangle PSA? (See Fig. 3).

Answer. The area of the triangle PSA is easy to calculate. It is enough to find the length of PS and then use Heron's formula, since we would know all three sides of the triangle.

Question 5. Are you happy with the final result for the problem you created?

Answer. Yes, to solve the problem it is necessary to try, and it is fairly "tricky". However, I did not manage to solve it using Euclidean Geometry. My professor has told me that solving a problem using Analytic Geometry leads to solutions which are neither smart, nor beautiful. I had to create a problem and make sure it has a solution. I hope someone else can find a prettier solution.

Question 6. How long did you need to create the problem and solve it?

Answer. I worked two days in total. I did not count specifically the number of hours during the two days, as I was not asked to do so from the beginning.

Question 7. To create the problem it seems that you did not use any software programs. Is this correct? Why is that?

Answer. I thought that the use of software programs during this task was forbidden (one of the "rules of the game"), even though I am quite familiar with Geogebra.

Question 8. If you were asked to generalize the problem you created, which generalization would you choose, whether you could solve the new problem or not?

Answer. As I wrote in the beginning, changing the square to a rectangle would be a generalization, but I think the steps in solving the problem would be the same. A "good" generalization would be to change the square to a cube! I think the corresponding problem would be considerably harder, perhaps as I am not particularly trained in solving problems with geometrical solids.

Research conclusions

Our first observation on the student's effort regarding problem posing is that they used a mixture of techniques, namely, modification of an initial problem, omission of data, and formulation of new statements. Hence, he has grasped fully the concept of problem posing.

Our second observation is that the student's effort on the creation of his problem includes many of the characteristics described in Kontorovich's [3] doctoral dissertation: "The transition from mathematical phenomena to the final problem is made using problem formulation techniques, some of which have been pointed out in past research: *reformulating, combining several problems, focusing on particular cases*. A previously unmentioned technique was *creating a misleading wrapper to the problem*. In this technique, the experts create wording which is supposed to create a stumbling block or unhelpful image of the problem for the solvers, and thus, to distance students from the problem's solution. In this way, the experts increase the cognitive difficulty of the problem and make it more novel and surprising for the solver" [3]. Even though the student is not yet an expert in problem posing, he tried hiding from the candidate solvers the "beginning" of the solution of his problem. This is a fundamental element for the creation of a "difficult" problem, as is the case of a problem of a competition.

The third observation is that the student was not asked to record the precise amount of time he needed to create the problem and to solve it. Furthermore, the student handed in to the researcher only the final exposition, and none of his drafts. We will take these research technicalities into consideration in new researches on problem posing, requiring more detailed information from the case studies.

Our fourth observation is on the complete understanding of the student on what exactly he was required to do, the care and attention he showed during this effort. He felt that he personally participated into an endeavor that was of great interest to him. From this point of view, our research succeeded in its goal, showcasing the capabilities of a student who makes an effort for a task he feels is personal. This point is in complete accord with the great researcher of the Didactics of Mathematics, Jeremy Kilpatrick, "Psychologists are fond of reminding us that a problem is not a problem for you until you accept it and interpret it as your own. One person cannot give a problem to another person; the second person has to construct the problem for himself or herself" [2, p. 124].

Our fifth observation – conclusion from the research conducted is that the student had fully comprehended the two problems on which he based his own problem. He found these two initial problems worthwhile and interesting. I assume that the choice of one or more problems as a foundation is a fundamental criterion for the successful creation of a new problem. The student, in addition to creating a difficult problem, assumed as an implied requirement of the research that he should solve it. The student's attitude expresses an essential connection between problem posing and problem solving. We need to mention that the researchers Mamona-Downs & Downs [5] had emphasized the following: "It was noticeable that students often posed problems without any regard to how to solve them; the actual issue implicit in the problems posed rarely strayed from those previously suggested by the conductors of the session; and though the given system was often adjusted, there were no cases of students posing problems in other systems" [5, p. 393]. This does not occur in our case study.

Our sixth observation is that the student did not use software programs to create his problem. It seems that "strong" students regard the use of software as a weakness especially in problem posing. Of course, it is known that making good use of software programs in problem posing is not only common practice among grown-up creators, but also a necessary tool.

From this research, even though it is a single case study, we draw the conclusion that the abilities of the gifted students not only in problem posing, but also in problem solving, exceed the expected result. We agree with Nolte [6] that new questions arise constantly, "What are the characteristics of problems suitable for challenging gifted students? Can these kinds of problems be used in regular classroom settings? Should problems be posed as complex problems or should special cognitive components of problem solving be trained isolated?" [6, p. 383], questions that need answers.

Of course, the road of extrapolation of definite conclusions for the role played by problem posing in the mathematical education of gifted students, and of typical students is still long. At this point we agree with the conclusion by Singer et al., "Just as there was a cry some years ago for problem solving to become an integral part of mathematics classrooms, we believe that this research forum on problem posing is needed to stimulate further discussion and research about ways in which problem posing can become a more natural and integral part of mathematics classrooms at all levels" [10, p. 138].

REFERENCES

[1] J. Cai & V. Cifarelli, *Exploring mathematical exploration: How do two college students formulate and solve their own mathematical problems?*, FOCUS on Learning Problems in Mathematics, **27** (3) (2005), 43–72.

[2] J. Kilpatrick, *Problem formulating: Where do good problems come from?* In: A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education*, (pp. 123–147). Hillsdale, NJ: Erlbaum, 1987.

[3] I. Kontorovich, *How do experts pose problems for mathematics competitions?* Research Thesis in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy. (Abstract). The Technion – Israel Institute of Technology.

[4] I. Kontorovich & B. Koichu, *A case study of an expert problem poser for mathematics competitions*, International Journal of Science and Mathematics Education, **14** (2016), 81–99

[5] J. Mamona-Downs & M. Downs, *The identity of problem solving*, Journal of Mathematical Behavior **24** (2005), 385–401.

[6] M. Nolte, K. Pamperien & K. Vorhlter, *Research and development tasks within the framework of the prima-projekt in Hamburg*, In: M. Nolte (Ed.), *Including the Highly Gifted and Creative Students – Current Ideas and Future Directions*, Proceedings of the 11th International Conference on Mathematical Creativity and Giftedness (MCG 11). Universität Hamburg, Germany, 2019.

[7] A. Poulos, *A research on the creation of problems for mathematics competitions*, The Teaching of Mathematics, **XX**, 1 (2017), 26–36.

[8] A. Poulos & J. Mamona-Downs, *Gifted students approaches when solving challenging mathematical problems*, In: M. Singer (Ed.), *Activities for, and Research on, Mathematically Gifted Students*, pp. 309–342, Springer, 2018.

[9] N. Robinson, *Two wrongs do not make a right: sacrificing the needs of gifted students does not solve society's unsolved problems*, Journal for the Education of the Gifted **26**, 4 (2003), 251–273.

[10] F.M. Singer, N. Ellerton, J. Cai & E. Leung, *Problem posing in mathematics learning and teaching: A research Agenda*, In: B. Ubuz, B. (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 137–166, Ankara, Turkey: PME, 2011.

[11] A. Soifel (Ed.), *Competitions for young mathematicians*, Perspectives from five continents. ICME-13 Monographs, Springer, 2017.

Department of Mathematics, Aristotle University of Thessaloniki, Mykonou 11, 54638, Thessaloniki, Greece

E-mail: andremath.poulos@gmail.com