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GENERALIZED ASSOCIATIVE AND COMMUTATIVE LAWS

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Abstract. School algebra is not an abstract mathematical discipline but its variables denote the numbers from a number system which is under elaboration. Basic properties of a number system are in the same time basic rules of algebra. Which other rules have to be deduced is a matter of concern to those who research problems of teaching algebra in school.

The commutative and associative laws are often seen formulated in math school books. But they have a full effect when they are used to define numerical value of sums and products of three and more members and when it is proved that this value is independent of the way how summands and factors are associated and ordered. To accept these proofs with understanding, students have to be prepared for deductive reasoning and to be acquainted with all different ways how elements of a set can be ordered and with the method of mathematical induction as well. Hence, it is a student of a serious secondary school of age 14 or more.

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1. Introduction

The essence of our approach to the study of number systems is the focus on the questions of transfer of properties of natural numbers to the systems of integers, positive rational, rational and real numbers. As a starting point, we take the basic operative properties of the system of natural numbers. These properties, when taken abstractly, as the axioms of a structure $\{S, +, \cdot, <\}$, where S is a non-empty set, "+", and "·" are two binary operations and "<" is an order relation, define a structure that is called the ordered semifield (See [1] and [2]). The system of natural numbers and all its extensions – systems of integers, positive rational, rational and real numbers are examples of the ordered semifield.

Thus, proceeding more concretely, a property deduced from the basic operative properties of the system of natural numbers is transferable to all above mentioned systems (the Peacock's principle of permanence) or, reasoning more abstractly, a property proved to be valid in the ordered semifield will be the property of all systems which are examples of the ordered semifield. In this note we will be occupied with the properties of sums and products of three or more numbers (members of the ordered semifield) deduced from the associative and commutative laws.

In school mathematics number systems are elaborated and rounded off in this and that way. Operative properties are sometimes founded on a solid intuitive ground, the other time just imposed by means of repeated exercises. Early algebra has its earliest stage in the period of learning natural numbers, then the periods of positive rational, rational and real numbers follow. In each of these periods dealing with the corresponding operative properties using literal expressions is the early algebra at that stage. We also hope that our approach of discussing number systems will throw some light on the problems of learning algebra in school.

Of all basic operative properties of number systems, those which are most frequently and explicitly formulated in school books are the commutative and associative laws for addition and multiplication. In primary books these laws are seen to be used to ease calculation. In the upper classes, children learn arithmetic and algebra doing more and more exercises. So they operate with sums and products of three and more numbers changing the order of summands and factors and associating them freely. With time, routine overcomes and the question if such manipulations are legitimate never raises.

In the case of natural numbers, the dependence of the idea of such numbers on the experience of sets at the sensory level and, in particular, independence of that idea of number of the way how elements of sets are grouped (Cantor principle) is a solid intuitive ground upon which the meaning and the properties of sums of three and more numbers are spontaneously acquired. Let us also add that when a property is induced intuitively, then there is no way to ensure its transfer from one number system to another. All in all, in school mathematics sums and products of three and more numbers are learnt by drill, while commutative and associative laws, when formulated, mainly figure as decorative attachments.

2. Significance of commutative and associative laws

Now we concentrate on discussing sums (and the way of dealing with products is carried out analogously). When addition is strictly conceived as a binary operation, then the expressions $(a+b)+c$ and $a+(b+c)$ have a precise meaning, whereas $a+b+c$ does not. The former two expressions indicate summing of three numbers by associating them two by two. The latter expression is called the sum of three numbers, but such a term has a clear syntactic meaning without determining the numerical value of that sum. This value is defined (taken) to be $(a+b)+c$, having so the order of summands and the way of their association fixed. But, as we are going to check it, the value of the sum of three numbers is independent of both, the order of summands and the way of their association.

Indeed, applying alternately commutative and associative laws, we obtain the following series of equalities:

$$\begin{aligned}(a+b)+c &= a+(b+c)=a+(c+b)=(a+c)+b=(c+a)+b\\ &= c+(a+b)=c+(b+a)=(c+b)+a=(b+c)+a\\ &= b+(c+a)=b+(a+c)=(b+a)+c.\end{aligned}$$

As we see there are six different ways of ordering summands (six permutations of the letters a, b and c). To each of these orderings, two ways of association of summands match. Altogether, it is twelve different ways of addition of two by two summands which produce one and the same number – the value of this sum. Therefore the sum of three numbers also has a precise semantic meaning being the number which is its value and which is simply denoted writing $a+b+c$.

Now let us consider the sum of four numbers $a_1+a_2+a_3+a_4$ taking that its numerical value is $s=(a_1+a_2+a_3)+a_4$ and let us also notice that each way of association of summands of this sum ends as $(a_1+a_2+a_3)+a_4$, $(a_1+a_2)+(a_3+a_4)$ and $a_1+(a_2+a_3+a_4)$. Being

$$(a_1+a_2+a_3)+a_4=((a_1+a_2)+a_3)+a_4=(a_1+(a_2+a_3))+a_4$$

and

$$a_1+(a_2+a_3+a_4)=a_1+((a_2+a_3)+a_4)=a_1+(a_2+(a_3+a_4)),$$

we see that there are 5 different ways how these summands can be associated. Adding a_4 to each $a_{k(1)}+a_{k(2)}+a_{k(3)}$ of 6 differently ordered summands of $a_1+a_2+a_3$, the following 4 sums are obtained.

$$\begin{aligned} &a_{k(1)}+a_{k(2)}+a_{k(3)}+a_4, \quad &&a_{k(1)}+a_{k(2)}+a_4+a_{k(3)},\\ &a_{k(1)}+a_4+a_{k(2)}+a_{k(3)}, \quad &&a_4+a_{k(1)}+a_{k(2)}+a_{k(3)}. \end{aligned}$$

We see that there are 24 such sums and, therefore, $5\cdot 24$ ($=120$) different ways how the summands of the sum $a_1+a_2+a_3+a_4$ can be ordered and associated.

Applying successively the associative law the following equalities result:

$$\begin{aligned} s&=(a_1+a_2+a_3)+a_4=((a_1+a_2)+a_3)+a_4=(a_1+a_2)+(a_3+a_4)\\ &=a_1+(a_2+(a_3+a_4))=a_1+((a_2+a_3)+a_4)=a_1+(a_2+a_3+a_4), \end{aligned}$$

what proves that the summands of the sum $a_1+a_2+a_3+a_4$ can be associated in all ways and these ways preserve the value s.

Let now $a_{k(1)}+a_{k(2)}+a_{k(3)}$ be any permutation of summands of $a_1+a_2+a_3$. Then

$$\begin{aligned} s&=(a_{k(1)}+a_{k(2)}+a_{k(3)})+a_4=((a_{k(1)}+a_{k(2)})+a_{k(3)})+a_4\\ &=(a_{k(1)}+a_{k(2)})+(a_{k(3)}+a_4)=(a_{k(1)}+a_{k(2)})+(a_4+a_{k(3)})\\ &=((a_{k(1)}+a_{k(2)})+a_4)+a_{k(3)}=(a_{k(1)}+a_{k(2)}+a_4)+a_{k(3)}. \end{aligned}$$

Since the summands of the sum of three numbers can be permuted freely, we also have

$$s=(a_{k(1)}+a_4+a_{k(2)})+a_{k(3)}=(a_4+a_{k(1)}+a_{k(2)})+a_{k(3)}.$$

Therefore, the numerical value of the sum of four numbers is independent of the way how its summands are associated and ordered.

In the case of the sum of three numbers, the independence of its value of the order of summands and the way of their association has been verified directly. But in the case of four summands, the number of needed verifications discourages us to think of such an idea. So we had to sketch a proof of that fact which also reflects the way how the generalized associative and commutative laws are proved.

Before we proceed with formulation of these laws and their proving we fix the meaning of the notations that follow. Namely, in the case of the sum $a_1 + a_2 + a_3$, we put

$$A_1 = a_1, \quad A_1' = a_2 + a_3, \quad A_2 = a_1 + a_2, \quad A_2' = a_3$$

For $m > 3$ and $a_1 + \cdots + a_m$ we put $A_1 = a_1$, $A_1' = a_2 + \cdots + a_m$ and for $k = 2, \ldots, m-2$

$$\begin{aligned} A_k &= a_1 + \cdots + a_k, \quad (a_1 + \cdots + a_2 = a_1 + a_2) \\ A_k' &= a_{k+1} + \cdots + a_m, \quad (a_{m-1} + \cdots + a_m = a_{m-1} + a_m) \end{aligned}$$

and $A_{m-1} = a_1 + \cdots + a_{m-1}$, $A_{m-1}' = a_m$.

Now we formulate the generalized associative law and we prove it applying mathematical induction.

2.1. *For each $n \geqslant 3$, all possible associations of two by two summands of the sum $a_1 + \cdots + a_n$ produce one and the same number – the numerical value of this sum.*

Proof. The statement is true for the sums of three numbers. Let us suppose it is true for each sum of m numbers, when $m < n$. As it is customary, we will be using the same symbol $a_1 + \cdots + a_m$ to denote the sum of m numbers as well as its value.

By application of the associative law the following equalities hold true

$$((a_1 + \cdots + a_{n-1}) + a_n = (A_k + A_k') + a_n = A_k + (A_k' + a_n),$$

$k = 1, \ldots, n-2$. Therefore, we see that all $n-1$ final associations of the sum $a_1 + \cdots + a_n$ are equal to one and the same number – the numerical value of this sum.

Our next step is formulation and proving of the generalized commutative law.

2.2. *The value of the sum $a_1 + \cdots + a_n$ is not effected by permutation of its summands.*

Proof. The statement is true for the sums of three numbers. Let us suppose it is true in the case of sums $a_1 + \cdots + a_m$ for each $m < n$. Let $a_1 + \cdots + a_n$ be the sum of n numbers and let $s = (a_1 + \cdots + a_{n-1}) + a_n$. Let $a_{k(1)} + \cdots + a_{k(n-1)}$ be a permutation of the summands $a_1 + \cdots + a_{n-1}$. Then, according to the induction hypothesis, $s = (a_{k(1)} + \cdots + a_{k(n-1)}) + a_n$ and applying both associative and commutative laws, we also have

$$\begin{aligned} s &= ((a_{k(1)} + \cdots + a_{k(n-2)}) + a_{k(n-1)}) + a_n \\ &= (a_{k(1)} + \cdots + a_{k(n-2)}) + (a_{k(n-1)} + a_n) \\ &= (a_{k(1)} + \cdots + a_{k(n-2)}) + (a_n + a_{k(n-1)}) \\ &= ((a_{k(1)} + \cdots + a_{k(n-2)}) + a_n) + a_{k(n-1)} \end{aligned}$$

$$= (a_{k(1)} + \cdots + a_{k(n-2)} + a_n) + a_{k(n-1)}.$$

Relying on the induction hypothesis again, from the position of $(n-1)$-th summand of $(a_{k(1)}+\cdots+a_{k(n-2)}+a_n)+a_{k(n-1)}$, a_n can be placed in the position of 1st, 2nd, ..., $(n-2)$-th summand. Since in the case of all these positions, the corresponding sums have their numerical value equal to s, we have this statement proved.

At the end, let us note that these two statements are usually proved to be valid in the case of any binary operation which is associative and commutative. Thus, they are valid for multiplication and their formulations and the proofs are identical with the case of addition, only the addition sign "+" has to be replaced with the multiplication sign "·". Of course, our interest here is confined to number systems and the deduction of properties from the basic properties of natural numbers with 0. Then, the properties deduced in that way are transferable to all other number systems including the system of real numbers. All in all, let us remark that everything here is accommodated to the reader who faces these facts for the first time.

3. A somewhat skeptical conclusion

Problems of finding the sum of several numbers are found here and there in elementary school math books. A case of finding such a sum is known as the story of Gauss. Namely, Gauss as a boy in elementary school was often bored with the tasks that his teacher used to give to the class. One day, in order to keep Gauss busy, the teacher assigned him to add all numbers from 1 to 100. Gauss associated the first and the last number, the second and the second-last number and so on: $(1+100)+(2+99)+\cdots+(50+51)$ obtaining so 50 summands each equal to 101. So the sought sum is $50 \cdot 101 = 5050$. This quick solution amazed and impressed the teacher of Gauss.

Another very well-known example of the sum of several numbers, found in math books of elementary school, is Pythagoras formula for the sum of successive odd numbers: $1+3+\cdots+(2n-1) = n^2$. Let us recall how the formula is derived. Taking a square pattern of n rows of n dots and when the dots are grouped in two different ways:

1. First dot in the first row, 3 following dots arranged in the form of "right angle" and so on, finally $2n-1$ dots arranged in the form of "right angle". Altogether it is $1+3+\cdots+(2n-1)$ dots.
2. In each of n rows there are n dots, altogether it is $n \cdot n$ dots.

Two expressions denote the number of dots in the pattern, thereby they represent the same number.

On the blackboard in the Repin's painting Mental Arithmetic, calculation of the value of the expression $(10^2+11^2+12^2+13^2+14^2) : 365$ is assigned to the boys. What could be the train of thought:

1. $10^2+11^2+12^2+13^2+14^2 = 5\cdot 100+(20+40+60+80)+(1+4+9+16) = 500+200+30 = 730$

or else

2. $(12-2)^2 + (12-1)^2 + 12^2 + (12+1)^2 + (12+2)^2 = 5 \cdot 12^2 + 2(1+4) = 720 + 10 = 730$.

In the first case summands are permuted and associated as the brackets command it. In the second case, each of differences $12^2 - 2 \cdot 12 \cdot 2$ and $12^2 - 2 \cdot 12 \cdot 1$ is taken as a summand which is associated with $2 \cdot 12 \cdot 2$ and $2 \cdot 12 \cdot 1$ respectively, producing $2 \cdot 12^2$. Both these sums are specific case of the relation $(a-b)+b = a$, $(b \leqslant a)$ which is valid for all natural numbers a and b, when b is not greater than a. Indeed, putting $a - b = x$ and by interdependence of addition and subtraction, $a = x + b = (a-b) + b$. This relation is transferable from the system of natural numbers to other number systems and it is an instance of a rule of algebra "with reason". (For rules imposed by drill is said that they are without reason.) Let us also remark that in the system of integers, the difference $a - b$ is taken as the sum $a+(-b)$. Then, ny the associative law $(a-b)+b = (a+(-b))+b = a+((-b)+b) = a$.

A solid intuitive basis upon which sums of several natural numbers gain their meaning and properties is the generalized additive scheme. This scheme is the union of n disjoint sets $S_1, S_2, \ldots, S_n$ of cardinality $a_1, a_2, \ldots, a_n$, respectively. Then, cardinal number of the union is denoted by $a_1 + a_2 + \cdots + a_n$. (This scheme is an abstract model for all those concrete situations in which sums of three or more numbers are involved). But when a concept and its properties are derived upon an intuitive basis, then there is no way to transfer them to the extended number systems.

School algebra is not a branch of abstract algebra but its variables always denote numbers from a number system. Its basic rules are basic properties of the concerned system. In addition to these basic rules which other rules have to be deduced to serve as the rules for the transformation of literal expressions? An analysis relied upon our approach of study of numbers systems could provide an answer. Proving properties is a logical procedure par excellence and such an elaboration of algebra is feasible in good secondary schools and when student's cognitive development is appropriate (age 14 and more). What of it can be found earlier (elementary school) in early algebra? I do not know where to find an answer to this question.

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