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A SHORT PROOF OF HÖLDER'S INEQUALITY USING CAUCHY-SCHWARZ INEQUALITY

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Abstract. The aim of this note is a to give a new and short proof that the Hölder inequality is implied by the Cauchy-Schwarz inequality.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space (μ is a positive measure). For all measurable functions $f, g\colon \Omega \to \mathbb{C}$, we recall the Hölder's inequality:

$$\text{(H)} \qquad \int_\Omega |fg|\, d\mu \le \left(\int_\Omega |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int_\Omega |f|^q\, d\mu\right)^{\frac{1}{q}}, \quad \forall p, q \ge 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

If $p = q = 2$ then we obtain the Cauchy-Schwarz inequality:

$$\text{(C-S)} \qquad \int_\Omega |fg|\, d\mu \le \left(\int_\Omega |f|^2\, d\mu\right)^{\frac{1}{2}} \left(\int_\Omega |f|^2\, d\mu\right)^{\frac{1}{2}}.$$

Their discrete versions are respectively given by:

$$(\mathrm{H}_d) \qquad \sum_{i=1}^n |x_i y_i| \le \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n |y_i|^q\right]^{\frac{1}{q}} := \|x\|_p \|y\|_q,$$

and

$$(\mathrm{C\text{-}S}_d) \qquad \sum_{i=1}^n |x_i y_i| \le \left[\sum_{i=1}^n |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^n |y_i|^2\right]^{\frac{1}{2}} := \|x\|_2 \|y\|_2,$$

for all positive integers n and all vectors $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{K}^n$, where the field $\mathbb{K}$ is real or complex.

Obviously, we have $(H) \implies (C\text{-}S)$. It is natural to raise the question: does $(C\text{-}S)$ imply (H)?

There is a positive answer to this question. Indeed, the proof of this fact is already known in the literature but, often, through indirect implications. See, for instance, [4, 6, 7].

Many connections between classical discrete inequalities were studied in the book [7], where, in particular, the equivalence $(H)_d \iff (C\text{-}S)_d$ was deducted through several intermediate results.

A. W. Marshall and I. Olkin pointed out in their book [6] that the Cauchy-Schwarz inequality implies Lyapunov's inequality which itself implies the arithmetic-geometric mean inequality. The conclusions are that, in a sense, the arithmetic-geometric mean inequality, Hölder's inequality, the Cauchy-Schwarz inequality, and Lyapunov's inequality are all equivalent [6, p. 457].

In 2006, Y-C. Li and S-Y Shaw [5] gave a proof of Hölder's inequality by using the Cauchy-Schwarz inequality. Their method lies on the fact that the convexity of a function on an open and finite interval is equivalent to continuity and midconvexity.

In 2007, the equivalence between the integral inequalities (H) and $(C\text{-}S)$ was studied by C. Finol and M. Wójtowicz in [3]. They gave a proof that $(C\text{-}S)$ implies (H) by using density arguments and mathematical induction.

The aim of this note is to investigate a new method of proving that $(C\text{-}S)$ implies (H). A report concerning this method of proof was recently posted in [1]. We present a proof of this implication which is different from those made in [3] and [5]. Indeed, our proof will make use of a simple improvement of the well known Young's inequality and the Cauchy-Schwarz inequality.

Let a, b be two positive numbers and let $\alpha \in [0,1]$. We denote by $Y(\alpha)$ the Young's inequality:

$$a^\alpha b^{1-\alpha} \le \alpha a + (1-\alpha)b. \tag{Y(α)}$$

2. Proof of the implication: $(C\text{-}S) \Longrightarrow (H)$

We avoid the trivial cases, so we suppose that $1 < p, q$ with $1/p + 1/q = 1$. We suppose also that $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$.

By using Young's inequality $(Y(\frac{1}{p}))$, for all positive numbers a and b, we have:

$$ab = \left[(\sqrt{a}^{\,p})^{\frac{1}{p}}(\sqrt{b}^{\,q})^{\frac{1}{q}}\right]^2 \le \left[\frac{1}{p}\sqrt{a}^{\,p} + \frac{1}{q}\sqrt{b}^{\,q}\right]^2 = \frac{1}{p^2}a^p + \frac{1}{q^2}b^q + \frac{2}{pq}a^{\frac{p}{2}}b^{\frac{q}{2}}. \tag{2.1}$$

By setting $a = |f(x)|/\|f\|_p$ and $b = |g(x)|/\|g\|_q$ in the inequality (2.1), we obtain the following inequality:

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \le \frac{|f(x)|^p}{p^2\|f\|_p^p} + \frac{|g(x)|^q}{q^2\|g\|_q^q} + \frac{2}{pq}\frac{|f(x)|^{p/2}}{\|f\|_p^{p/2}}\frac{|g(x)|^{q/2}}{\|g\|_q^{q/2}}. \tag{2.2}$$

By integrating both sides of (2.2), we get

$$\int_\Omega \frac{|f(x)g(x)|}{\|f\|_p\|g\|_q}\,d\mu(x) \le \frac{1}{p^2} + \frac{1}{q^2} + \frac{2}{pq\|f\|_p^{p/2}\|g\|_q^{q/2}}\int_\Omega |f|^{p/2}|g|^{q/2}\,d\mu.$$

Therefore, we have

$$\int_\Omega |fg|\,d\mu \le \left(\frac{1}{p^2}+\frac{1}{q^2}\right)\|f\|_p\|g\|_q + \frac{2}{pq}\|f\|_p^{1-\frac{p}{2}}\|g\|_q^{1-\frac{q}{2}}\int_\Omega |f|^{p/2}|g|^{q/2}\,d\mu. \tag{2.3}$$

Now, by using the Cauchy-Schwarz inequality, we obtain the following inequality:

$$\int_\Omega |f|^{p/2}|g|^{q/2}\,d\mu \le \left[\int_\Omega |f|^p\,d\mu\right]^{\frac{1}{2}}\left[\int_\Omega |g|^q\,d\mu\right]^{\frac{1}{2}} = \|f\|_p^{\frac{p}{2}}\|g\|_q^{\frac{q}{2}}. \tag{2.4}$$

From (2.3) and (2.4), we deduce that

$$\int_\Omega |fg|\,d\mu \le \left(\frac{1}{p^2}+\frac{1}{q^2}+\frac{2}{pq}\right)\|f\|_p\|g\|_q = \left(\frac{1}{p}+\frac{1}{q}\right)^2\|f\|_p\|g\|_q = \|f\|_p\|g\|_q.$$

This finishes the proof.

Remark. The inequality (2.3) implies the following improvement to Hölder's inequality.

$$\int_\Omega |fg|\,d\mu \le \|f\|_p\|g\|_q\left(1-\frac{1}{pq}\left\|\frac{|f|^{\frac{p}{2}}}{\|f\|_p^{\frac{p}{2}}}-\frac{|g|^{\frac{q}{2}}}{\|g\|_q^{\frac{q}{2}}}\right\|_2^2\right), \tag{2.5}$$

for all $f \in L_p \setminus \{0\}$ and all $g \in L_q \setminus \{0\}$. The inequality (2.5) above was obtained by J. M. Aldaz [2] in a different manner.

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REFERENCES

[1] M.Akkouchi, *Cauchy-Schwarz inequality implies Hölder's inequality*, RGMIA Res. Rep. Coll. **21** (2018), Art. 48, 3p.

[2] J. M. Aldaz, *Self improvement of the inequality between arithmetic and geometric means*, J. Math. Inequalities, **3**, 2 (2009), 213–216.

[3] C. Finol, M. Wojtowicz, *Cauchy-Schwarz and Hölder's inequalities are equivalent*, Divulgaciones Matematicas **15**, 2 (2007), 143–147.

[4] C. A. Infantozzi, *An introduction to relations among inequalities*, Amer. Math. Soc. Meeting 700, Cleveland, Ohio 1972; Notices Amer. Math. Soc. **14** (1972), A819-A820, pp. 121–122.

[5] Yuan-Chuan Li, Sen-Yen Shaw, *A proof of Hölder's inequality using the Cauchy-Schwarz inequality*, J. Inequal. Pure and Appl. Math., **7** (2), Art. 62, 2006.

[6] A. W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York-London, 1979.

[7] D. S. Mtirinović, J. E. Pečarić, A. M. Fink, *Classical and New Inequalities in Analysis.* Kluwer Academic Publishers, 1993.

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