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NUMBER SYSTEMS CHARACTERIZED BY THEIR OPERATIVE PROPERTIES

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Abstract. In our paper *Structuring Systems of Natural, Positive Rational and Rational Numbers* [The Teaching of Mathematics 22, 1 (2019)], we have studied operative properties of number systems (i.e., the properties of operations and the order relation). In the same paper we have selected a number of operative properties of the system N of natural numbers with 0 which we called the *basic operative properties of* N.

Let $\{S,+,\cdot,<\}$ be a structure, where S is a non-empty set, "+", "·" are two binary operations and "<" is the order relation. We called provisionally such a structure N-structure, when its axioms are basic operative properties of N taken abstractly and we proved that *the system N of natural numbers with 0 is the smallest N-structure*.

Here we rename the N-structure and call it the *ordered semifield*. Adding to the axioms of the ordered semifield the axiom: $(\forall a)(\exists b)\, a+b=0$, then such a structure we call the *ordered semifield with additive inverse* and adding to the same axioms, the axiom: $(\forall a\neq 0)(\exists b)\, a\cdot b=1$, we call such a structure the *ordered semifield with multiplicative inverse*. When both of these axioms are added to the axioms of the ordered semifield, then such a system of axioms coincides with the axioms of the *ordered field*.

In this note we prove that *the system of integers is the smallest ordered semifield with additive inverse* and that *the system of positive rational numbers with 0 is the smallest ordered semifield with multiplicative inverse*.

The fact that the system of rational numbers is the smallest ordered field is well known. At the end of this note we also include a proof of this fact.

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1. Introduction

A good acquaintance with the properties of number systems is essential for planning and a proper didactical transforming of the content of school algebra. That is the reason why we include this kind of our research in education.

This note is a complement to our paper [1], where we have been concerned with operative properties of number systems (i.e. the properties of operations and the order relation). In the same paper, we have selected a number of these properties in the case of the system $\mathbb{N}$ of natural numbers with 0 and have called them the basic operative properties of the system N. We list these properties right below:

(i) $(\forall k)(\forall l)\, k+l=l+k$

(ii) $(\forall k)(\forall l)(\forall m)\,(k+l)+m=k+(l+m)$

(iii) $(\exists 0)(\forall k)\, k+0=k$

(iv) $(\forall k)(\forall l)\, kl=lk$

(v) $(\forall k)(\forall l)(\forall m)\,(kl)m=k(lm)$

(vi) $(\exists 1)(0<1$ *and* $(\forall k)\, k\cdot 1=k)$

(vii) $(\forall k)(\forall l)(\forall m)\, k(l+m)=kl+km$

(viii) $(\forall k)(\forall l)\,(k<l \iff (\exists m>0)\, k+m=l)$

(ix) $(\forall k)(\forall l)\,(k<l$ *or* $k=l$ *or* $l<k)$

(x) $(\forall k)(\forall l)(\forall m)$
$(k<l \iff k+m<l+m)$

(xi) $(\forall k)(\forall l)(\forall m>0)$
$(k<l \iff km<lm)$

(Variables are denoted by the letters from the middle of the alphabet).

In the same paper [1], we have verified that basic properties of $\mathbb{N}$ together with those deduced from them suffice to be a ground upon which extensions of $\mathbb{N}$ to the system $\mathbb{Q}_+$ of positive rational numbers with 0 and the system $\mathbb{Q}$ of rational numbers are constructed. Moreover, all basic properties of $\mathbb{N}$ and all those deduced from them are transferable to these extended systems. Hence, the selection of these properties makes the Peacock's principle of permanence precise because they are exactly the properties of $\mathbb{N}$ that are transferable to the extended systems.

Since the letters denoting variables in different systems also denote elements of different sets, it is reasonable to choose them to be specific for each system:

Number system	Letters denoting variables
$\mathbb{N}$ – natural numbers	$k, l, m, n, \dots$
$\mathbb{Z}$ – integers	$u, v, x, y, \dots$
$\mathbb{Q}_+$ – positive rational numbers	$q, r, s, t, \dots$
$\mathbb{Q}$ – rational numbers	$a, b, c, d, \dots$

Technically, the transfer of the properties is carried out simply by their transcribing, using the letters specific for the corresponding system.

Thus, in the case of the system $\{\mathbb{Z}, +, \cdot, <\}$ of integers, its basic operative properties are those of $\mathbb{N}$, where k, l, m are replaced by u, v, x, adding also the existence of additive inverse:

$$(\forall u)(\exists v)\ u+v=0.$$

Similarly, in the case of the system $\{\mathbb{Q}_+, +, \cdot, <\}$ of positive rational numbers with 0, its basic operative properties are those of $\mathbb{N}$, where k, l, m are replaced by q, r, s, adding to them the existence of multiplicative inverse:

$$(\forall q \neq 0)(\exists r)\ q\cdot r=1.$$

And finally, when basic properties of $\mathbb{N}$ are transcribed replacing k, l, m by a, b, c, adding the existence of both additive and multiplicative inverse, the basic operative properties of the system $\{\mathbb{Q}, +, \cdot, <\}$ of rational numbers are obtained.

2. A wider view of number systems

Abstracting, each number system can be viewed as a structure $\{S, +, \cdot, <\}$, where S is a non-empty set, "+" and "$\cdot$" are two binary operations and "<" is an order relation. When the properties of the system $\mathbb{N}$ are transcribed replacing k, l, m by a, b, c and when they are taken to be axioms of the structure $\{S, +, \cdot, <\}$, then we call such a structure *ordered semifield* (and in [1], it was called $\mathbb{N}$-structure). The examples of the ordered semifield are the systems of natural numbers, of integers, of positive rational numbers and of rational and real numbers.

When to the axioms of the ordered semifield, the axiom that ensures the existence of additive inverse: $(\forall a)(\exists b)\ a + b = 0$ is added, then we call such a structure *ordered semifield with additive inverse.* In this case, the corresponding examples are the system of integers and of rational and real numbers.

When to the axioms of the ordered semifield, the axiom that ensures the existence of multiplicative inverse: $(\forall a \neq 0)(\exists b)\ a \cdot b = 1$ is added, then we call such a structure *ordered semifield with multiplicative inverse* (and in [1], it was called $\mathbb{Q}_+$-structure). In this case, the corresponding examples are the systems of positive rational numbers with 0, of rational numbers and of real numbers.

When to the axioms of the ordered semifield, the axioms that ensure the existence of both additive and multiplicative inverse are added, then such a system of axioms coincides with the axioms of ordered field and, of course, a structure satisfying this extended system of axioms is called *ordered field.* Among the examples of the ordered field, the systems of rational and real numbers are of particular interest for us.

Now let $\{S, +, \cdot, <\}$ be an ordered semifield. Let us define a mapping $f\colon \mathbb{N} \to S$, taking $f(0) = 0_S = a_0$, $f(1) = 1_S = a_1$. Let us suppose that if $f(n)$ has already been defined, then we take $f(n+1) = a_n + 1_S = a_{n+1}$. In [1], the following equalities have been proved: $a_{n+m} = a_n + a_m$ and $a_{n\cdot m} = a_n \cdot a_m$, as well as the implication $n < m \implies a_n < a_m$.

Thus, we see that $f[\mathbb{N}]$ is an isomorphic image of $\mathbb{N}$, what we can express in the following way: *The system* $\mathbb{N}$ *of natural numbers with* 0 *is the smallest ordered semifield.* This is, of course, a characterization of the system $\mathbb{N}$ via its operative properties.

3. Further characterization of number systems

For further characterization of number systems via their operative properties, we will be using the mappings which are the extensions of the mapping f defined in the preceding section and its properties mentioned there.

1. *The system of integers* $\{\mathbb{Z}, +, \cdot, <\}$ *is the smallest ordered semifield with additive inverse.*

Proof. Let $\{S, +, \cdot, <\}$ be an arbitrary ordered semifield with additive inverse and let $f\colon \mathbb{N} \to S$, $f(n) = a_n$ be the mapping described above. We can extend it to a mapping $g\colon \mathbb{Z} \to S$ in the following way:

$$g(z) = \begin{cases} a_z, & \text{for } z \geqslant 0, \\ -a_{-z}, & \text{for } z < 0. \end{cases}$$

Let us prove that this mapping is injective and that it preserves operations "+" and "$\cdot$" and relation "<".

1° Let $z_1, z_2 \in \mathbb{Z}$ and $g(z_1) = g(z_2)$. It is clear that there are just the following two possibilities: $z_1, z_2 \geqslant 0$ or $z_1, z_2 < 0$. In the first case, we have that $a_{z_1} = a_{z_2}$, and, since the mapping $f\colon \mathbb{N} \to S$ is injective, it follows that $z_1 = z_2$. In the second case, from $-a_{-z_1} = -a_{-z_2}$ it follows that $a_{-z_1} = a_{-z_2}$, hence $-z_1 = -z_2$ and $z_1 = z_2$. Thus, g is an injective mapping.

2° Let us show that $g(z_1 + z_2) = g(z_1) + g(z_2)$ holds for all $z_1, z_2 \in \mathbb{Z}$. This relation holds trivially when $z_1 = 0$ or $z_2 = 0$. We distinguish further the following cases:

(a) $z_1, z_2 > 0$. Then $g(z_1 + z_2) = a_{z_1+z_2} = a_{z_1} + a_{z_2} = g(z_1) + g(z_2)$.

(b) $z_1 > 0$, $z_2 < 0$, $z_1 + z_2 \geqslant 0$. Then the following holds:

$$g(z_1 + z_2) = a_{z_1+z_2} = a_{z_1-(-z_2)} = a_{z_1} - (a_{-z_2}) = g(z_1) + g(z_2).$$

(c) $z_1 > 0$, $z_2 < 0$, $z_1 + z_2 < 0$. Then

$$\begin{aligned} g(z_1 + z_2) &= -a_{-(z_1+z_2)} = -a_{(-z_2)-z_1} = -(a_{-z_2} - a_{z_1}) = a_{z_1} + (-a_{-z_2}) \\ &= g(z_1) + g(z_2). \end{aligned}$$

(d) $z_1, z_2 < 0$. Then we have

$$\begin{aligned} g(z_1 + z_2) &= -a_{-(z_1+z_2)} = -a_{(-z_1)+(-z_2)} = -(a_{-z_1} + a_{-z_2}) \\ &= -a_{-z_1} + (-a_{-z_2}) = g(z_1) + g(z_2). \end{aligned}$$

3° Let us show that $g(z_1 \cdot z_2) = g(z_1) \cdot g(z_2)$ holds for all $z_1, z_2 \in \mathbb{Z}$. Again, this holds trivially when $z_1 = 0$ or $z_2 = 0$. Consider the following cases.

(a) $z_1, z_2 > 0$. Then $g(z_1 \cdot z_2) = a_{z_1 \cdot z_2} = a_{z_1} \cdot a_{z_2} = g(z_1) \cdot g(z_2)$.

(b) $z_1 > 0$, $z_2 < 0$ (i.e., $z_1 \cdot z_2 < 0$). Then we have

$$\begin{aligned} g(z_1 \cdot z_2) &= -a_{-(z_1 \cdot z_2)} = -a_{z_1 \cdot (-z_2)} = -(a_{z_1} \cdot a_{-z_2}) \\ &= a_{z_1} \cdot (-a_{-z_2}) = g(z_1) \cdot g(z_2). \end{aligned}$$

(c) $z_1, z_2 < 0$ (i.e., $z_1 \cdot z_2 > 0$). Then the following holds

$$g(z_1 \cdot z_2) = a_{z_1 \cdot z_2} = a_{(-z_1) \cdot (-z_2)} = (-a_{-z_1}) \cdot (-a_{-z_2}) = g(z_1) \cdot g(z_2).$$

4° Finally, we show that $z_1, z_2 \in \mathbb{Z}$ and $z_1 < z_2$ imply $g(z_1) < g(z_2)$.

From $z_1 < z_2$ it follows that $z_1 + m = z_2$ holds for some $m > 0$. As it has been proved in 2°, $g(z_2) = g(z_1) + g(m)$ holds, hence $g(z_1) < g(z_2)$ because $g(m) > 0$.

Thus, we have proved that $g[\mathbb{Z}]$ is an isomorphic copy of $\mathbb{Z}$.

2. *The system of positive rational numbers with* 0, $\{\mathbb{Q}_+, +, \cdot, <\}$ *is the smallest semifield with multiplicative inverse.*

Proof. Let $\{S, +, \cdot, <\}$ be an arbitrary ordered semifield with multiplicative inverse and let $f\colon \mathbb{N} \to S$, $f(n) = a_n$ be the mapping described above. Let us define the mapping $h\colon \mathbb{Q}_+ \to S$ by

$$h(k/l) = a_k : a_l, \qquad \text{for } k, l \in \mathbb{N},\ l > 0.$$

We show that this mapping is injective and that it preserves operations "+" and "·" and relation "<".

1° Let $k_1/l_1, k_2/l_2 \in \mathbb{Q}_+$ be such that $h(k_1/l_1) = h(k_2/l_2)$. Then $a_{k_1} : a_{l_1} = a_{k_2} : a_{l_2}$. It follows that $a_{k_1} \cdot a_{l_2} = a_{k_2} \cdot a_{l_1}$, hence $a_{k_1 l_2} = a_{k_2 l_1}$. This means that $k_1 l_2 = k_2 l_1$ and finally $k_1/l_1 = k_2/l_2$. Thus, h is an injective mapping.

2° For arbitrary elements k_1/l_1, k_2/l_2 of $\mathbb{Q}_+$ the following holds:

$$\begin{aligned} h(k_1/l_1 + k_2/l_2) &= h((k_1 l_2 + k_2 l_1)/(l_1 l_2)) = a_{k_1 l_2 + k_2 l_1} : a_{l_1 l_2} \\ &= (a_{k_1} a_{l_2} + a_{k_2} a_{l_1}) : (a_{l_1} a_{l_2}) = (a_{k_1} : a_{l_1}) + (a_{k_2} : a_{l_2}) \\ &= h(k_1/l_1) + h(k_2/l_2). \end{aligned}$$

3° For all k_1/l_1, $k_2/l_2 \in \mathbb{Q}_+$ we have that

$$\begin{aligned} h((k_1/l_1) \cdot (k_2/l_2)) &= h((k_1 k_2)/(l_1 l_2)) = a_{k_1 k_2} : a_{l_1 l_2} \\ &= (a_{k_1} a_{k_2}) : (a_{l_1} a_{l_2}) = (a_{k_1} : a_{l_1}) \cdot (a_{k_2} : a_{l_2}) \\ &= h(k_1/l_1) \cdot h(k_2/l_2). \end{aligned}$$

4° Let k_1/l_1 i k_2/l_2 be elements of $\mathbb{Q}_+$ such that $k_1/l_1 < k_2/l_2$. Then $k_1 l_2 < k_2 l_1$, and it follows that $a_{k_1 l_2} < a_{k_2 l_1}$. Further, $a_{k_1} a_{l_2} < a_{k_2} a_{l_1}$, and finally $a_{k_1} : a_{l_1} < a_{k_2} : a_{l_2}$, i.e., $h(k_1/l_1) < h(k_2/l_2)$.

Thus, we have proved that $h[\mathbb{Q}_+]$ is an isomorphic copy of $\mathbb{Q}_+$.

The following statement is well known and can be found in courses of mathematical analysis. Here, we also include our proof of it.

3. *The system of rational numbers* $\{Q, +, \cdot, <\}$ *is the smallest ordered field.*

Proof. Let $\{S, +, \cdot, <\}$ be an arbitrary ordered field and let $g\colon \mathbb{Z} \to S$ be the mapping used in the proof of assertion **1**. Let us define the mapping $j\colon \mathbb{Q} \to S$ by

$$j(p/q) = g(p) : g(q), \qquad \text{for } p \in \mathbb{Z},\ q \in \mathbb{N},\ q \neq 0.$$

We will prove that j is an injective mapping, and that it preserves operations "+" and "·" and relation "<".

1° Let $p_1/q_1, p_2/q_2 \in \mathbb{Q}$ be such that $j(p_1/q_1) = j(p_2/q_2)$. Then $g(p_1) : g(q_1) = g(p_2) : g(q_2)$. Or else $g(p_1) \cdot g(q_2) = g(p_2) \cdot g(q_1)$, what implies that $g(p_1q_2) = g(p_2q_1)$. This means that $p_1q_2 = p_2q_1$ and hence $p_1/q_1 = p_2/q_2$. This proves that j is an injective mapping.

2° The following holds for arbitrary elements p_1/q_1, p_2/q_2 in $\mathbb{Q}$:

$$\begin{aligned} j(p_1/q_1 + p_2/q_2) &= j((p_1q_2 + p_2q_1)/(q_1q_2)) = g(p_1q_2 + p_2q_1) : g(q_1q_2) \\ &= (g(p_1)g(q_2) + g(p_2)g(q_1)) : (g(q_1)g(q_2)) \\ &= (g(p_1) : g(q_1)) + (g(p_2) : g(q_2)) = j(p_1/q_1) + j(p_2/q_2). \end{aligned}$$

3° Let $p_1/q_1, p_2/q_2 \in \mathbb{Q}$; then:

$$\begin{aligned} j((p_1/q_1) \cdot (p_2/q_2)) &= j((p_1p_2)/(q_1q_2)) = g(p_1p_2) : g(q_1q_2) \\ &= (g(p_1)g(p_2)) : (g(q_1)g(q_2)) = (g(p_1) : g(q_1)) \cdot (g(p_2) : g(q_2)) \\ &= j(p_1/q_1) \cdot j(p_2/q_2). \end{aligned}$$

4° Let p_1/q_1 and p_2/q_2 be elements of $\mathbb{Q}$ satisfying $p_1/q_1 < p_2/q_2$. Then $p_1q_2 < p_2q_1$, implying that $g(p_1q_2) < g(p_2q_1)$. It follows that $g(p_1)g(q_2) < g(p_2)g(q_1)$ and finally $g(p_1) : g(q_1) < g(p_2) : g(q_2)$, i.e., $j(p_1/q_1) < j(p_2/q_2)$.

Thus, we have proved that $j[\mathbb{Q}]$ is an isomorphic copy of $\mathbb{Q}$.

At the end, let us formulate the Peacock's principle of permanence in terms of here defined concepts: A property of the ordered semifield is also the property of all structures which are its examples and, in particular, of the systems of natural numbers with 0, of integers, of positive rational numbers with 0, rational numbers as well as of the system of real numbers.

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