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CONCEPTUAL TASKS IN MATHEMATICS EDUCATION

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Abstract. The article examines conceptual tasks in both mathematics teaching and research in mathematics education. It presents a number of conceptual task types and suggests several research directions that may be pursued in years to come.

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1. Conceptual tasks in mathematics teaching

Is mathematics a skill-oriented school subject? The answer is undoubtedly "yes" since traditional mathematics teaching mainly cultivates skills, neglecting conceptual understanding of the underlying domain (see [5]; cf. [6]). These skills are primarily fostered through solving procedural tasks involving fully quantified objects, which students often solve by using appropriate remembered rules (algorithms) without knowing why they work. Having in mind the distinction between process-based and object-based thinking [10], it seems that for the majority of students mathematics education only promotes process-based thinking. As a result, even in an able class, only a few students may achieve object-based thinking. In other words, by using Skemp's terms ([12]), instrumental understanding of mathematics still dominates over relational understanding of this subject.

Object-based thinking can effectively be assessed by using conceptual tasks involving objects that are not (fully) quantified, such as the following tasks for earlier secondary education:

1. Which number is bigger:

a) $a+5$ or $4+a$ (a is a whole number)?

b) $\dfrac{a}{b}$ or $\dfrac{2a}{3b}$ (a and b are natural numbers)?

c) x^{1997} or x^{2000} (x is a real number)? Consider all cases.

2. Prove that the sum of the distances from any point in the interior of an equilateral triangle to its sides is equal to the length of the triangle's altitude.

3. When is a number of the form '$aabbba$' (a and b are some digits) divisible by 6?

4. A store offers a discount, but you must pay some sales tax. Which would be calculated first, discount or tax? Explain your answer.

5. Find the radius of a circle inscribed in a triangle if its perimeter and area are known.

6. A mountaineer started his trip at 8:00 a.m. arriving at a mountain hut in the evening. Having spent the night there, the mountaineer started down the next day at 9:00 a.m. using the same trail. Is there a point on the trail where he was at the same place at the same time in both days? Give an explanation.

7. A cylindrical rod weights 9 kilos. What would be the weight of another rod (made of the same material) which is two times thicker and two times shorter than this one?

8. Some workers finished one task in a certain time. When will another task, two times less in scope than that one, be finished within three times shorter time if the number of workers is increased by 4?

9. Interpret the following distance vs. time graph.

Such tasks can also deal with general assertions regarding mathematical objects and their structural properties, requiring the solver to determine whether the assertions are true. For example:

10. Are the following assertions true:

a) in any triangle, the sum of two altitudes is greater than the third one;

b) two rhombi are always similar;

c) the product of two irrational numbers is always an irrational number;

d) for any real number x, $x^2 + 1$ is greater than x.

11. Which of the following assertions are true:

a) every non-constant geometric progression is either increasing or decreasing;

b) any divergent sequence is unbounded;

c) there is a function defined on the entire real axis which is even and odd at the same time;

d) the derivative of an odd differentiable function is an even function.

Despite the high educational value of conceptual tasks, mathematical literature is indeed lacking in them. This is particularly true in highly proceduralized areas such as equation solving. As a result, students are rarely required to solve the following or a similar task:

12. Supposing that $f(x)$ and $g(x)$ are two real functions, do the following equations $f(x) = g(x)$ and $\dfrac{1}{f(x)} = \dfrac{1}{g(x)}$ have the same solutions? Explain your answer.

If we accept the following proceduralo-conceptual task distinction according to which procedural tasks rely heavily on computations that do not necessarily require understanding of the underlying domain, whereas conceptual tasks that involve very little computations do require this understanding, several types of conceptual tasks may be introduced. Thus apart from tasks on not (fully) quantified objects, we may deal with other conceptual tasks such as tasks on objects equivalence, tasks on solution existence (non-existence) and tasks on erroneous arguments.

- Tasks on objects equivalence deal with identical functions and equivalent equations (inequalities) like task 12. Other examples are:

13. Are some of the following functions identical?

$$f(x) = x, \quad g(x) = \sqrt{x^2}, \quad h(x) = \frac{x^2}{x} \quad \text{and} \quad p(x) = (\sqrt{x})^2.$$

Give an explanation. (Think of the precise definition of a function.)

14. Which of the following equations have the same solutions (a and b may be expressions of any kind)?

$$ab = 0, \quad \frac{a}{b} = 0 \quad , a\sqrt{b} = 0, \quad \frac{a}{\sqrt{b}} = 0 \quad \text{and} \quad \frac{\sqrt{a}}{b} = 0.$$

Explain your answer.

- Tasks on solution existence (non-existence) require the solver to determine whether the given task has a solution, such as:

15. Whether the following equation (inequality) has a solution?

a) $1 + x^2 = \sin x$,

b) $\sqrt{x-2} + \sqrt{1-x} < \sqrt{x}$,

c) $1 + \ln x = \cos x$.

Give an explanation.

Of course, such tasks are not bounded to equation (inequality) solving, e.g.,

16. Two diagonally opposite corners of a chessboard are removed. Can the remaining board be covered with dominoes covering two squares each.

- Tasks on erroneous arguments present lines of reasoning yielding absurd results and the solver is required to find out why these lines are wrong. For example:

17. What is wrong with the argument:

a) $2x + 1 = 3x$, $2x - 2 = 3x - 3$, $2(x-1) = 3(x-1)$, $2 = 3$ by cancellation?

b) $x > \frac{1}{x}$, $x^2 > 1$, $x > 1$ or $x < -1$; for $x = -2$, $x > \frac{1}{x}$ yields $-2 > -.5$?

c) $i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$?

d) $\sqrt{x^3} = x\sqrt{x}$; for $x = -1$, we obtain $i = -i$?

e) $\int \frac{dx}{x} = \int \frac{1}{x}\,dx = \frac{1}{x}\,x - \int x\left(-\frac{1}{x^2}\right)dx = 1 + \int \frac{dx}{x}$, thus $1 = 0$?

18. Squaring a circle of radius r is still possible by using the right circular cylinder of radius r and altitude $r/2$. Since its lateral surface is equal to the area of the circle, we can paint this surface, then roll the cylinder on a paper to produce the rectangle of the same area, and finally transform this rectangle into the square of the same area. Why is this argument wrong?

Although tasks 13–18 or similar mostly deal with fully quantified objects, they indeed require little computation, and it is mainly conceptual understanding of the underlying domain which is relevant to success in solving them. It is therefore reasonable to consider these tasks as conceptual.

It is important to note that conceptual tasks are not bounded to the types introduced above. There are other tasks that are also conceptual in nature. These are, for example, tasks on simple function transformations such as:

19. The function $y = f(x)$ is given graphically. Sketch the graphs of the following functions: $y = f(a - x)$ and $y = f(2x) - b$.

It seems that this or a similar task can be considered as an instance of tasks on object transformations. Let us recall that this kind of conceptual tasks is also encountered in geometry problems of the following type "Cut figure F1 into some pieces and use these to compose figure F2" like:

20. Cut a square into four pieces and compose a right trapezoid out of them.

2. Conceptual tasks in mathematics education research

Surprisingly enough, it was only recently realized that mathematics education also needs to be based upon conceptual tasks as they, contrary to traditional procedural tasks, can fully assessed whether genuine understanding of the underlying domain is really achieved (see [2]). Although this study only examines calculus requiring that "Calculus must change from a skill-oriented course to a concept oriented course." and that "Technical skills can be thrown overboard, but conceptual skills stressed and developed more than previously." (pp. 63, 65), there is no doubt that this request is relevant to mathematics education in general. It is particularly true today when computer-based mathematics education is available. This is because computer can be used to introduce a new balance of instructional time by decreasing the time for procedural skills and increasing the time for conceptual understanding [3], which seems to promote better understanding [8, 9].

Mathematics educators frequently make a distinction between procedural and conceptual knowledge (i.e., between algorithmic performance and understanding [7]), assuming that procedural knowledge represents knowledge regarding "a sequence of actions", whereas conceptual knowledge manifests knowledge that is "rich in relationships" [4]. As "pure forms of either type of knowledge are seldom, if ever, exhibited" [11, p. 183], it seems that the proceduralo-conceptual knowledge distinction may effectively be addressed by using the proceduralo-conceptual task distinction. Note that although this knowledge distinction, which has been accepted as both general and fundamental, has opened up an important research area, there is in general no consensus so far on an adequate theoretical model of these type of knowledge and their relation [1].

Having in mind the previous discussion, further research regarding conceptual tasks may be primarily directed towards the following goals:

- creating sets of conceptual tasks for different mathematical topics, especially for those that are highly procedural in nature;
- developing effective (computer-based) teaching methods dealing with solving conceptual tasks;
- examining whether solving conceptual tasks can be proceduralized;
- devising a suitable model relating to the proceduralo-conceptual knowledge distinction and operationalizing it by appropriate sets of procedural and conceptual tasks;
- developing effective (computer-based) teaching methods regarding the acquisition and coordination of procedural and conceptual knowledge;
- uncovering and refining cognitive and affective factors that influence the acquisition and coordination of procedural and conceptual knowledge, by examining different mathematical contents, various modes of teaching, different sort of students, etc.

If we opt for mathematics education that is less procedurally oriented, research into the issues listed above is still needed.

3. Final remark

Traditional physics education is also mainly procedurally oriented, but it is easier to find conceptual tasks in physical than in mathematical literature. While physics has its "flying circus"—a six-hundred problem collection [13], a mathematical flying circus may only be under development as, to our knowledge, a book on systematized mathematical conceptual tasks according to task types, grades or areas has not been published so far. We hope that this article will inspire the reader to collect, classify, devise and use conceptual tasks in his/her own teaching and research, resulting in the formation of a flying mathematical circus in years to come.

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