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STRICT MONOTONICITY OF NONNEGATIVE STRICTLY CONCAVE FUNCTION VANISHING AT THE ORIGIN

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Abstract. In this paper we prove that every nonnegative strictly concave function on the unbounded closed interval $[0, +\infty)$ is strictly increasing, provided it vanishes at the origin. With the help of this result, we then show that the strict monotonicity condition of the theorem concerning the metric transforms is redundant. We also provide a companion version of this result for merely concave nonnegative function which vanishes only at the origin.

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1. Introduction

It is well-known that if (X, d) is a metric space, then the function $d'\colon X \times X \to \mathbb{R}$ defined by $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a new metric on X, for reference, see [1, 3, 6]. Since the function $f\colon [0, +\infty) \ni x \mapsto \frac{x}{1+x}$ is nonnegative and strictly concave on $[0, +\infty)$ and vanishes at $x = 0$, many authors generalized the metric d' as follows: If (X, d) is a metric space and $f\colon [0, +\infty) \to \mathbb{R}$ is a nonnegative increasing function which is strictly concave, and vanishes at $x = 0$, then the composition $f \circ d$ of d followed by f is a new metric on X. See references [1–5]. If f is such a mapping, then we call f a *metric transform* on X, see [4] for details. In [6], the author demands that f is continuous, so as to be a metric transform on X.

In this article, we show that the increasing monotonicity condition (or the continuity condition) of such a generalization is actually redundant. As Theorem 1 below shows, $f \circ d$ is a metric transform on X, provided $f\colon [0, +\infty) \to \mathbb{R}$ is nonnegative strictly concave, with $f(0) = 0$, because such a condition guarantees, as we prove, that f is automatically strictly increasing. We also prove a companion version (that is, Theorem 2) for merely concave function which is nonnegative and vanishes only at the origin.

This paper is organized as follows. In Section 2 we prove our two theorems, for strictly concave function and merely concave function, respectively. In Section 3, we first give the corresponding corollaries concerning the metric transforms. Then we use the result to consider the usual metric transforms.

2. Main results

Recall the definition of (strictly) concave function of a real variable.

Definition 1. Let I be a nondegenerate interval in $\mathbb{R}$. We say that a function $f\colon I \to \mathbb{R}$ is concave (*downward*) on I, if for all x and y in I, and all $\lambda \in (0,1)$, the condition

$$f\big(\lambda x + (1-\lambda)y\big) \geq \lambda f(x) + (1-\lambda) f(y)$$

holds. We say that f is strictly concave (*downward*) on I, if for all x and y in I, with $x \neq y$, and all $\lambda \in (0,1)$, the condition

$$f\big(\lambda x + (1-\lambda)y\big) > \lambda f(x) + (1-\lambda) f(y)$$

holds.

Our main theorem is the following.

Theorem 1. *Let $f\colon [0,+\infty) \to \mathbb{R}$ be strictly concave on $[0,+\infty)$. If $f(x) \geq 0$ for all $x \in [0,+\infty)$ and $f(0)=0$, then the following hold:*

(i) *f is strictly subadditive on $(0,+\infty)$, that is, for all $0<a<b$, we have*

$$f(a+b) < f(a) + f(b).$$

(ii) *For all $x>0$, $f(x)>0$.*

(iii) *f is strictly increasing on $[0,+\infty)$.*

Proof. Let $f\colon [0,+\infty) \to \mathbb{R}$ satisfy the given conditions.

(i) Let $0<a<b$. Then $0<a+b$. By strict concavity of f, together with $f(0)=0$, we deduce that

$$\begin{aligned} f(a) &= f\left(\frac{a}{a+b}\cdot(a+b) + \frac{b}{a+b}\cdot 0\right) \\ &> \frac{a}{a+b} f(a+b) + \frac{b}{a+b} f(0) = \frac{a}{a+b} f(a+b), \end{aligned} \tag{1}$$

and

$$\begin{aligned} f(b) &= f\left(\frac{b}{a+b}\cdot(a+b) + \frac{a}{a+b}\cdot 0\right) \\ &> \frac{b}{a+b} f(a+b) + \frac{a}{a+b} f(0) = \frac{b}{a+b} f(a+b). \end{aligned} \tag{2}$$

By adding up relations (1) and (2), we arrive at

$$f(a+b) < f(a) + f(b), \qquad \text{provided } 0<a<b. \tag{3}$$

(ii) Suppose that, contrary to what has to be shown, there exists $x'>0$ such that $f(x') \leq 0$. Since f is assumed to be nonnegative, we have $f(x')=0$. Thus, from strict subadditivity of f on $(0,+\infty)$,

$$f(2x') = f(x'+x') < f(x') + f(x') = 2f(x') = 0,$$

contradicting that f is nonnegative.

(iii) Suppose that f is not strictly increasing on $[0, +\infty)$. Then there exist x_1, x_2 with $0 \le x_1 < x_2$, such that $f(x_1) \ge f(x_2)$. Since $x_2 > 0$, by Part (ii) of this theorem, we have $f(x_2) > 0$, and hence $f(x_1) > 0$, which implies that $x_1 > 0$, for $f(0) = 0$.

Then there are two cases to consider: $f(x_1) = f(x_2)$, or else $f(x_1) > f(x_2)$. If $f(x_1) = f(x_2)$, then, for every real number x_3 such that $x_3 > x_2$, we have

$$x_2 = \frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3.$$

From strict concavity of f, it follows that

$$\begin{aligned} f(x_1) = f(x_2) &= f\left(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3\right) \\ &> \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3), \end{aligned}$$

which leads to

$$\frac{x_2 - x_1}{x_3 - x_1} f(x_1) > \frac{x_2 - x_1}{x_3 - x_1} f(x_3),$$

and so

$$f(x_3) < f(x_2) = f(x_1), \qquad \text{for all } x_3 \text{ with } x_1 < x_2 < x_3.$$

If $f(x_1) > f(x_2)$, then, for every real number x_3 with $x_3 > x_2$, we have

$$x_2 = \frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3.$$

By strict concavity of f, we have

$$\begin{aligned} f(x_2) = f\left(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3\right) &> \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3) \\ &> \frac{x_3 - x_2}{x_3 - x_1} f(x_2) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3), \end{aligned}$$

which leads to

$$\frac{x_2 - x_1}{x_3 - x_1} f(x_2) > \frac{x_2 - x_1}{x_3 - x_1} f(x_3),$$

and so

$$f(x_3) < f(x_2), \qquad \text{for all } x_3 \text{ with } x_1 < x_2 < x_3.$$

Thus, we have shown that if there are two real numbers x_1 and x_2 such that $0 < x_1 < x_2$ with $f(x_1) \ge f(x_2)$, then, for each real number x_3 with $x_3 > x_2$, $f(x_3) < f(x_2)$.

Now fix some x_3 such that $x_1 < x_2 < x_3$. Then we have

$$f(x_3) < f(x_2). \tag{4}$$

Set $\delta = \dfrac{f(x_2) - f(x_3)}{2} > 0$. Then, by (4),

$$f(x_3) + \delta = f(x_3) + \frac{f(x_2) - f(x_3)}{2} = \frac{f(x_2) + f(x_3)}{2} < f(x_2). \tag{5}$$

Take

$$z = \frac{f(x_2)(x_3 - x_2)}{\delta} + x_2 + x_3. \tag{6}$$

Since $x_2 > 0$, from Part (ii) it follows that $f(x_2) > 0$, and so

$$z > x_3 > x_2. \tag{7}$$

Since $x_3 = \frac{z - x_3}{z - x_2} x_2 + \frac{x_3 - x_2}{z - x_2} z$, we deduce that

$$\begin{aligned} f(x_3) &= f\left(\frac{z - x_3}{z - x_2} x_2 + \frac{x_3 - x_2}{z - x_2} z\right) \\ &> \frac{z - x_3}{z - x_2} f(x_2) + \frac{x_3 - x_2}{z - x_2} f(z). \end{aligned} \tag{8}$$

Combining (5) and (8), together with (7), we arrive at

$$f(x_2) > f(x_3) + \delta > \frac{z - x_3}{z - x_2} f(x_2) + \frac{x_3 - x_2}{z - x_2} f(z) + \delta,$$

implying that $\frac{x_3 - x_2}{z - x_2} f(x_2) > \frac{x_3 - x_2}{z - x_2} f(z) + \delta$, and thus

$$f(x_2) - \frac{z - x_2}{x_3 - x_2}\delta > f(z). \tag{9}$$

Now inserting (6) into the left-hand side of (9), we have

$$\begin{aligned} f(x_2) - \frac{z - x_2}{x_3 - x_2}\delta &= f(x_2) - \frac{\frac{f(x_2)(x_3 - x_2)}{\delta} + x_2 + x_3 - x_2}{x_3 - x_2}\delta \\ &= f(x_2) - f(x_2) - \frac{x_3}{x_3 - x_2}\delta = -\frac{x_3}{x_3 - x_2}\delta < 0. \end{aligned}$$

Consequently, it follows from (9) that

$$0 > f(x_2) - \frac{z - x_2}{x_3 - x_2}\delta > f(z),$$

which contradicts that f is nonnegative on $[0, +\infty)$.

Therefore we have proved that f is strictly increasing on $[0, +\infty)$. ∎

By a similar argument, we provide a companion version of Theorem 1, where f is merely concave on $[0, +\infty)$, as follows.

THEOREM 2. *Let $f\colon [0, +\infty) \to \mathbb{R}$ be concave on $[0, +\infty)$. If $f(0) = 0$ and $f(x) \geq 0$ for all $x \in (0, +\infty)$, then the following hold:*

(α) *f is subadditive on $[0, +\infty)$, that is, for all $0 \leq a < b$, we have*

$$f(a + b) \leq f(a) + f(b).$$

(β) *f is increasing on $[0, +\infty)$.*

Proof. Let $f\colon [0,+\infty) \to \mathbb{R}$ satisfy the given conditions.

(α) Let $0 \le a < b$. Then $0 < a+b$. By concavity of f, together with $f(0)=0$, we deduce that

$$\begin{aligned} f(a) &= f\left(\frac{a}{a+b}\cdot(a+b) + \frac{b}{a+b}\cdot 0\right) \\ &\ge \frac{a}{a+b}f(a+b) + \frac{b}{a+b}f(0) = \frac{a}{a+b}f(a+b), \end{aligned} \tag{10}$$

and

$$\begin{aligned} f(b) &= f\left(\frac{b}{a+b}\cdot(a+b) + \frac{a}{a+b}\cdot 0\right) \\ &\ge \frac{b}{a+b}f(a+b) + \frac{a}{a+b}f(0) = \frac{b}{a+b}f(a+b). \end{aligned} \tag{11}$$

By adding up (10) and (11), we arrive at

$$f(a+b) \le f(a) + f(b), \qquad \text{provided } 0 \le a < b.$$

(β) Suppose that f is not increasing on $[0,+\infty)$. Then there exist x_1, x_2 with $0 \le x_1 < x_2$, such that $f(x_1) > f(x_2)$. Since $x_2 > 0$, by hypothesis, we have $f(x_2) \ge 0$ and hence $f(x_1) > 0$, which implies that $x_1 > 0$, for $f(0) = 0$.

Now since $f(x_1) > f(x_2)$, for every real number x_3 with $x_3 > x_2$, we have

$$x_2 = \frac{x_3 - x_2}{x_3 - x_1}x_1 + \frac{x_2 - x_1}{x_3 - x_1}x_3.$$

By concavity of f, we have

$$\begin{aligned} f(x_2) &= f\left(\frac{x_3 - x_2}{x_3 - x_1}x_1 + \frac{x_2 - x_1}{x_3 - x_1}x_3\right) \ge \frac{x_3 - x_2}{x_3 - x_1}f(x_1) + \frac{x_2 - x_1}{x_3 - x_1}f(x_3) \\ &> \frac{x_3 - x_2}{x_3 - x_1}f(x_2) + \frac{x_2 - x_1}{x_3 - x_1}f(x_3), \end{aligned}$$

which leads to

$$\frac{x_2 - x_1}{x_3 - x_1}f(x_2) > \frac{x_2 - x_1}{x_3 - x_1}f(x_3),$$

and so

$$f(x_3) < f(x_2), \qquad \text{for all } x_3 \text{ with } x_1 < x_2 < x_3.$$

Now fix some x_3 such that $x_1 < x_2 < x_3$. Then we have

$$f(x_3) < f(x_2). \tag{12}$$

Set $\delta = \dfrac{f(x_2) - f(x_3)}{2} > 0$. Then, by (12),

$$f(x_3) + \delta = f(x_3) + \frac{f(x_2) - f(x_3)}{2} = \frac{f(x_2) + f(x_3)}{2} < f(x_2). \tag{13}$$

Take

$$z = \frac{f(x_2)(x_3 - x_2)}{\delta} + x_2 + x_3. \tag{14}$$

Since $x_3 > x_2 > 0$, from hypothesis it follows that $f(x_2) \geq 0$, and we get that

$$z > x_3 > x_2. \tag{15}$$

Since $x_3 = \frac{z - x_3}{z - x_2} x_2 + \frac{x_3 - x_2}{z - x_2} z$, we deduce that

$$\begin{aligned} f(x_3) &= f\left(\frac{z - x_3}{z - x_2} x_2 + \frac{x_3 - x_2}{z - x_2} z\right) \\ &\geq \frac{z - x_3}{z - x_2} f(x_2) + \frac{x_3 - x_2}{z - x_2} f(z). \end{aligned} \tag{16}$$

Combining (13) and (16) together with (15) we arrive at

$$f(x_2) > f(x_3) + \delta \geq \frac{z - x_3}{z - x_2} f(x_2) + \frac{x_3 - x_2}{z - x_2} f(z) + \delta,$$

implying that

$$\frac{x_3 - x_2}{z - x_2} f(x_2) > \frac{x_3 - x_2}{z - x_2} f(z) + \delta,$$

and thus

$$f(x_2) - \frac{z - x_2}{x_3 - x_2} \delta > f(z). \tag{17}$$

Now inserting (14) into the left-hand side of inequality (17), we have

$$\begin{aligned} f(x_2) - \frac{z - x_2}{x_3 - x_2} \delta &= f(x_2) - \frac{\frac{f(x_2)(x_3 - x_2)}{\delta} + x_2 + x_3 - x_2}{x_3 - x_2} \delta \\ &= f(x_2) - f(x_2) - \frac{x_3}{x_3 - x_2} \delta = -\frac{x_3}{x_3 - x_2} \delta < 0. \end{aligned}$$

Consequently, it follows from (17) that

$$0 > f(x_2) - \frac{z - x_2}{x_3 - x_2} \delta > f(z),$$

which contradicts the fact that f is nonnegative on $[0, +\infty)$.

Therefore we have proved that f is increasing on $[0, +\infty)$. ■

3. Applications

The above theorems show that the monotonicity property is implied from concavity. Thus, for a function to be metric transform, there is no need to check its monotonicity, as the following corollaries show.

Corollary 1. *Suppose that* $f\colon [0,+\infty) \to [0,+\infty)$ *is strictly concave and satisfies* $f(0)=0$. *Let* (X,d) *be a metric space. Define function* d_f *on* $X \times X$ *by*

$$d_f(x_1,x_2) = f\big(d(x_1,x_2)\big)$$

for all $x_1, x_2 \in X$. *Then* d_f *is also a metric on* X.

Proof. If $x_1 = x_2$, then, since d is a metric on X, we have $d(x_1,x_2)=0$, and so $d_f(x_1,x_2) = f\big(d(x_1,x_2)\big) = f(0) = 0$. Conversely, suppose that $d_f(x_1,x_2)=0$, that is, $f\big(d(x_1,x_2)\big)=0$. Then we must have $d(x_1,x_2)=0$, for otherwise we would have $d(x_1,x_2)>0$, and thus, by Part (ii) of Theorem 1, it leads to $d_f(x_1,x_2) = f\big(d(x_1,x_2)\big) > 0$, a contradiction. Hence we have shown that $d_f(x_1,x_2)=0$ if and only if $x_1 = x_2$.

Since as a metric, d is symmetric, and so is d_f.

We continue to show that d_f satisfies the triangle inequality on X. Let $x_1, x_2, x_3 \in X$. Then

$$d(x_1,x_3) \le d(x_1,x_2) + d(x_2,x_3).$$

By Theorem 1, f is strictly increasing on $[0,+\infty)$, and as a result, from the strict subadditivity (3) of f, we have

$$\begin{aligned} d_f(x_1,x_3) = f\big(d(x_1,x_3)\big) &\le f\big(d(x_1,x_2)+d(x_2,x_3)\big) \\ &\le f\big(d(x_1,x_2)\big) + f\big(d(x_2,x_3)\big) \\ &= d_f(x_1,x_2) + d_f(x_2,x_3). \end{aligned}$$

Therefore we have proved that d_f is a new metric on X, provided d is a metric on X. ∎

By a similar argument we can prove, by Theorem 2, the following corollary.

Corollary 2. *Suppose that* $f\colon [0,+\infty) \to \mathbb{R}$ *is concave and satisfies* $f(0)=0$, *and* $f(x)>0$ *for all* $x \in (0,+\infty)$. *Let* (X,d) *be a metric space. Define function* d_f *on* $X \times X$ *by*

$$d_f(x_1,x_2) = f\big(d(x_1,x_2)\big)$$

for all $x_1, x_2 \in X$. *Then* d_f *is also a metric on* X.

Now consider the following functions

$$f\colon [0,+\infty) \ni x \mapsto \sqrt{x}, \tag{18}$$

$$f\colon [0,+\infty) \ni x \mapsto \frac{x}{1+x}, \tag{19}$$

$$f\colon [0,+\infty) \ni x \mapsto \min\{x,1\}. \tag{20}$$

By Corollary 1, the functions in (18) and (19) are metric transforms. Since it is easy to show that the function (20) is concave on $[0,+\infty)$, (see [3] for reference), by Corollary 2, the function (20) is also a metric transform.

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