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A SIMPLE PROOF OF THE CHANGE OF VARIABLE THEOREM FOR THE RIEMANN INTEGRAL

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Abstract. In this note, we present a simple proof of the change of variable theorem for the Riemann integral. The proof gives an alternate approach of how a class material on this topic can sistematically be carried out through a simple, succinct, and convenient way.

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Despite the intricacies, most authors use elementary approaches to prove the change of variable theorem for the Riemann integral. Some of the proofs, either for the whole or a part of the theorem, can be seen, e.g. in [1,4–8,10–12] (in [1], Bagby even demonstrated that a non-measure-theoretic argument works for a part of the theorem for functions taking values in a Banach space). In this note, by contrast, we demonstrate a simple method that gives a less elementary, yet much simpler, proof of the theorem.

Recall that a real-valued function f on $X \subseteq \mathbb{R}$ is said to be *Lipschitz* if there exists a positive number M such that for all $s, t \in X$, $|f(s) - f(t)| \leq M|s - t|$. A set A of real numbers is said to have *a measure zero* if for any $\epsilon > 0$ there exists a countable collection $\{(u_n, v_n)\}_{n=1}^{\infty}$ of open intervals such that $A \subseteq \bigcup_{n=1}^{\infty}(u_n, v_n)$ and $\sum_{n=1}^{\infty}(v_n - u_n) < \epsilon$. Two functions f and g on $[a, b]$ are said to be *equal almost everywhere*, which we write $f = g$ a.e., if the set $\{x \in [a, b] : f(x) \neq g(x)\}$ is of measure 0.

We assume that all integrability are in the Riemann sense. We shall prove the theorem by resorting to the following well-known properties:

(i) [Fundamental Theorem of Calculus of the First Form] If $F\colon [a, b] \to \mathbb{R}$ is Lipschitz and $F' = f$ a.e., for some integrable function f on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ (see [3, Theorem 2]).

(ii) [Fundamental Theorem of Calculus of the Second Form] If $h\colon [a, b] \to \mathbb{R}$ is integrable and is continuous at x_0, then the function $\psi(x) := \int_a^x h(t)\,dt$, $x \in [a, b]$, is differentiable at x_0 with $\psi'(x_0) = h(x_0)$ (see, e.g. in [2, Theorem 7.3.5]).

(iii) [Lebesgue Criterion for Riemann Integrability] A function f is integrable on $[a, b]$ if and only if f is bounded and continuous a.e. on $[a, b]$.

(iv) Given a function $F\colon [a,b] \to \mathbb{R}$ having a finite derivative F' on $X \subseteq [a,b]$, then $F(X)$ is of measure 0 if and only if $F' = 0$ a.e. on X (see, e.g. in [9, p. 515]).

Note that, as in the proof of Theorem 11 of [10], here we also use property (iv) but with a different presentation. Unlike his argument there which partly relies on the change of variable theorem for Lebesgue integral (whose proof is derived from that of the Henstock-Kurzweil integral), ours here entirely relies within the Riemann theory. This provides an alternate approach of how a teaching material on this topic can sistematically be presented in a simple, succinct, and convenient way.

One usual form of the change of variable theorem for the Riemann integral is as follows.

Theorem 1. *Let f be Riemann integrable on $[a,b]$, and for some $c \in [a,b]$, $F(x) := \int_c^x f(t)\,dt$, $x \in [a,b]$. Then $(g \circ F)f$ is Riemann integrable on $[a,b]$ if and only if g is Riemann integrable on $J := F([a,b])$. In either case we have*

$$\int_a^b g(F(x))f(x)\,dx = \int_{F(a)}^{F(b)} g(y)\,dy.$$

First notice that properties (i), (ii), and (iii), along with the additivity of integrals over subintervals, give the following fact: given an integrable function f on $[a,b]$, then a function F on $[a,b]$ is Lipschitz and $F' = f$ a.e. if and only if, for some constant $c \in [a,b]$, $F(x) = \int_c^x f(t)\,dt$. The change of variable theorem can then also be stated as follows.

Theorem 2. *Suppose that $F\colon [a,b] \to \mathbb{R}$ is Lipschitz, $F' = f$ a.e. for some Riemann integrable f on $[a,b]$, and $g\colon f([a,b]) \to \mathbb{R}$ is bounded. Then $(g \circ F)f$ is Riemann integrable on $[a,b]$ if and only if g is Riemann integrable on $J := F([a,b])$. In either case we have*

$$\int_a^b g(F(x))f(x)\,dx = \int_{F(a)}^{F(b)} g(y)\,dy.$$

We need the following lemma to prove Theorem 2.

Lemma 3. *Let F and f be as in Theorem 2, and*

$$A := \{x \in [a,b] : f \text{ is continuous at } x \text{ and } F'(x) = f(x)\}.$$

Then for any $x \in A$, $(g \circ F)f$ is continuous at x if and only if $f(x) = 0$ or g is continuous at $F(x)$.

Proof. Let $x \in A$. We first show part ($\Rightarrow$). Suppose that $(g \circ F)f$ is continuous at x and $f(x) \neq 0$. We wish to show that g is continuous at $F(x)$. Since f is continuous at x, there exists a subinterval $[c,d]$ on which x is an interior point, $|f| \geq \delta$ for some $\delta > 0$, and f has the same sign. Since by property (i),

$F(z_1) - F(z_2) = \int_{z_1}^{z_2} f(x)\,dx$, it follows that $|F(z_1) - F(z_2)| \geq \delta|z_2 - z_1|$, for all $z_1, z_2 \in [c,d]$. This implies that F is one-to-one on $[c,d]$. Since $(g \circ F)f$ and f are both continuous at x, and $f(x) \neq 0$, it follows from the property for the limit of a quotient that $\lim_{z\to x} g(F(z)) = g(F(x))$. Write $y = F(z)$, for any $z \in [c,d]$. Since $F|_{[c,d]}$ is one-to-one, it follows that $y \to F(x)$ if and only if $z \to x$, and hence

$$\lim_{y\to F(x)} g(y) = \lim_{z\to x} g(F(z)) = g(F(x)).$$

Thus g is continuous at $F(x)$. For part ($\Leftarrow$), if $f(x) = 0$, as f is continuous at x and g is bounded, it follows from the squeeze principle that $(g \circ F)f$ is continuous at x. Also, if g is continuous at $F(x)$, since F is continuous at x, it follows that $g \circ F$ is continuous at x, and therefore since f is continuous at x, while noting the property for the limit of a product, we conclude that $(g \circ F)f$ is continuous at x. ∎

Proof of Theorem 2. Let A be as in Lemma 3, $B := \{x \in [a,b] : f(x) \neq 0\}$ and

$$C := \{x \in [a,b] : g \text{ is discontinuous at } F(x)\}.$$

Since f is integrable and $F' = f$ a.e., it follows from property (iii) that $[a,b] \setminus A$ is of measure 0. Also, in view of property (iii), g is integrable on J if and only if $F(C)$ is of measure 0. These facts, along with Lemma 3 and property (iv), gives

$$\begin{aligned} (1) \qquad (g \circ F)f \text{ is continuous a.e. on } [a,b] &\iff A \cap (B \cap C) \text{ is of measure } 0 \\ &\iff B \cap C \text{ is of measure } 0 \\ &\iff F' = f = 0 \text{ a.e. on } C \\ &\iff F(C) \text{ is of measure } 0 \\ &\iff g \text{ is continuous a.e. on } J. \end{aligned}$$

This proves the first part of the theorem. Suppose now $(g \circ F)f$ and g are both integrable, i.e. bounded and continuous a.e. by property (iii). Let

$$\phi(x) := \int_a^x g(F(t))f(t)\,dt \quad \text{and} \quad \theta(x) := \int_{F(a)}^{F(x)} g(y)\,dy \quad (x \in [a,b]).$$

Then $\phi' = (g \circ F)f$ a.e. by property (ii). Since g is bounded, $\theta' = 0 = (g \circ F)f$ on $[a,b] \setminus B$. By the chain rule for derivatives, $\theta' = (g \circ F)f$ on $B \setminus C$. Since $B \cap C$ is of measure 0 by (1), it follows that $\theta' = (g \circ F)f$ a.e. Thus $\phi' = (g \circ F)f = \theta'$ a.e. Since ϕ and θ are both Lipschitz, it follows from property (i) that $\phi(b) = \int_a^b g(F(t))f(t)\,dt = \theta(b)$. ∎

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