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STRUCTURING THE SUBJECT MATTER OF ARITHMETIC I

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Abstract. In the case of construction of the block of numbers up to 100 (block N_{100}), all processes that lead from observation to the creation of abstract concepts are traced and didactically shaped. Sums with summands in the block N_{20} having the value exceeding 20 are used to extend this block to the block N_{100}. Then addition and subtraction of two-digit numbers is treated and, for the sake of understanding, all intermediate steps are expressed in words and symbols. But when these operations are performed automatically these steps are suppressed and the expressing in words is reduced to its inner speech contraction.

The block N_{100} is a natural frame within which multiplication is introduced and where the multiplication table is built up. In the school practice the meaning of multiplication is established through examples of situations having the structure of a finite family of finite equipotent sets which we call multiplicative scheme. Some suitable models of multiplicative scheme (as, for example, boxes with marbles) are used to establish main properties of multiplication. Let us also add that we build multiplication table grouping its entries according to the way how the corresponding products are calculated and we find that these ways of calculation should be learnt, instead of learning this table by rote.

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1. Introduction

This paper is planned to be one in a series intended to be a contribution to the improvement in didactics of primary school arithmetic—construction of the set N of natural numbers, endowed with four operations and algorithms for doing them and the relation "larger than", including also the main properties of these operations and this relation. Having its high utilitarian value, arithmetic has always been an important subject of elementary education. In the days that are gone, its content was learnt in a formalistic, scholastic way—numbers were introduce by interpretation of their decimal notations with a constant emphasis on place value, while the operations were learnt as formal procedures performed on these notations. A learner of such arithmetic had to view its content as an area of knowledge hardly connected with mathematical contents that followed it.

A number of decades before the "New Math" started, search for meaning brought some new ideas of teaching arithmetic. And with some exaggeration, we could say that during the "New Math" period sets were ousting numbers from arithmetic. But what is the state of affairs of today's school arithmetic is probably much harder to say and we have no intention to do it. Some references that are included throw light on directions in which the teaching of arithmetic has been changing and developing.

A century long discourse on issues of learning and teaching arithmetic, combined with our personal experience that we gained as teachers of educational institutions and authors of primary school textbooks on mathematics makes us join those who take a firm stand that arithmetic has also a high developmental value and should be elaborated in the way to be a basic mathematical subject whose content permeates all other contents of school mathematics. To achieve this goal the subject matter of arithmetic must be deliberately structured, tracing all processes leading from observation to the creation of abstract concepts. Without having clear views how these processes are didactically shaped, the subject matter of arithmetic reduces to the mere number facts which are situated outside of the learning situations.

Let us recall that numbers are quite abstract concepts formed in the mind of the learner in their natural dependence on discrete realities—sets of visible things. This act of forming is the interplay between observation and intelligent activities through which the observed material is processed. Things to be observed are selected and the mental operations which arise out of these activities are spotted as inner representations of set theoretic operations. The activities of observing are related to the ways of expressing them in words and in symbols that is to say by means of thinking. A verbal description of cognitive functions and symbolic codification of empiric facts is a key for a deeper understanding of the processes of learning arithmetic which could neither follow from various theorizing matters nor from a good knowledge of mathematics at higher levels.

Although our paper [11] has its focus on early algebra, it also contains a complete explanation of the ways how blocks of numbers, first up to 10 and then up to 20, are structured. Thus the present paper can be considered as a continuation of the paper [11] and its main objective will be structuring of the block of numbers up to 100 (block N_{100}). In doing it, we have to carry out the following tasks:

- to construct the block N_{100} as an extension of the block N_{20},
- to elaborate addition and subtraction of two digit numbers, displaying all steps of these procedures when understanding is concerned and suppressing them when automatic performance is the aim,
- to display various examples of the multiplicative scheme, followed by a multiplication task as the way of establishing meaning of multiplication as an abstract operation,
- to construct the multiplication table,
- to establish general properties of multiplication and to express them rhetorically and symbolically.

Let us note that the other tasks related to this block will be carried out in a paper that is planned to follow this one.

We have no fear that the scientific level of this paper will be lowered by a sketch of elaboration of the concrete didactical units of school arithmetic that flows throughout it. We also hope that, elaborating this concrete content, some new ideas will be contributed, but its main purpose is to be a basis upon which our general considerations touch the ground.

2. Block of numbers up to 100 as an additive structure

To build a block of numbers means to achieve all didactical tasks attached to it. Besides of it, each number of that block gets a unique decimal notation and an iconic representation that is suitable for quick recognition. Here we suppose that the operation of addition has been established as an abstract operation when the block N_{10} (and later the block N_{20}) was formed by observation (or imagination) of the examples of additive scheme and attaching of corresponding sums. For example, when one set has 8 elements and the other one 6, then they together with their union make an example of additive scheme and the sum $8+6$ is attached to denote the number of elements of this union. Then, relied upon iconic representation and counting (or simple calculation by crossing the ten line), the sum $8 + 6$ is equated with its decimal notation 14. Therefore, a difference should be made between addition as an abstract operation which has a permanent meaning, not being changed when going from a block to a larger one, while equating sums with their decimal notation is a special technical task which is just an instance of a much larger program that we call *decimalization*.

Building the block of numbers up to 20, the sums as, for example, $12+14$, $20+10$, etc. have their meaning established in the frame of this block. But, intentionally, the equating of these sums with their decimal notations is avoided, because such notations do not belong to this block. Hence, sums are used to generate numbers before than their decimal notation is given as a result of regrouping elements of two sets related to the summands.

There are two usually practiced ways to start with this extension. One is the introduction of tens: thirty is 3 tens, denoted by 30; forty is 4 tens, denoted by 40; ... ; hundred is 10 tens, denoted by 100. One reason for such a start is the fact that addition and subtraction of tens is analogous to the performance of these operations with units in the block N_{10}. The other is connected with the idea of extension of the process of counting. Twenty one is $20 + 1$, denoted by 21; twenty two is $20 + 2$, denoted by 22; ... ; thirty is $20 + 10$, denoted by 30. Similarly all other groups of ten numbers up to 100 are introduced. All these numbers have also their iconic representation as it is shown in Fig. 1 in case of numbers 26 and 42.

Representing iconically 26, it is seen that it consists of 2 tens and 6 units and 42 of 4 tens and 2 units. To emphasize the place value, two digits are colored differently one is red and the other one blue. The colors of digits are in accordance with the colors of arrangements which represent units and tens (the blue is on left of the red).

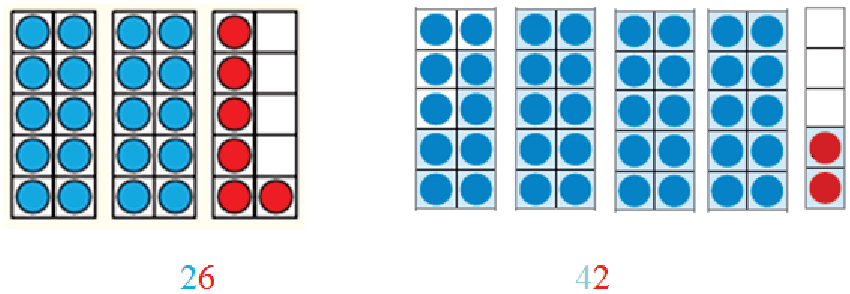


Fig. 1. Number pictures representing two-digit numbers

The number pictures that we use are arrangements of small disks (counters). A sub-arrangement representing 10, we call 10-arrangement and those representing units are called similarly according to the number of units. For example, describing arrangements in Fig. 1, we say that the first consists of 2 10-arrangements and one 6-arrangement and the second of 4 10-arrangements and one 2-arrangement.

Let us note that formerly, in a number of our papers as well as in some school books that we have written, we used the number pictures consisted of signs being thick lines (sticks) which cross over each other when the numbers 6, 7, ... , 10 are represented. Some teachers who used these books reported that the children were apt to see such arrangements as a whole (being the picture of a lattice) and they had some difficulty to single out these lines as the signs representing the unit of counting. In the paper [11] six requirements are formulated that a type of number pictures should satisfy. Now we add here an additional one: *Signs representing the unit of counting should be clearly seen as the constituent parts of number pictures.*

Typical exercises to learn place value are as the following ones

(a) *The number* 48 *consists of* __ *tens and* __ *units. Say how you imagine the number picture which represents* 48.

(b) *The number* 80 *consists of* __ *tens and* __ *units. Say how you imagine the number picture which represents* 80.

Etc.

The simplest cases of addition in this block are equating of sums of tens and units to their decimal notation ($70+3=73$, $80+9=89$, etc.). Illustrating tens by their number pictures, their addition and subtraction goes analogously to performance of these operations on units in the block N_{10}. Exercises of this type are:

$$4+3=7,\quad 40+30=\_\_\_;\qquad 5+4=\_\_,\quad 50+40=\_\_\_;\qquad \text{etc.}$$
$$7-4=\_\_,\quad 70-40=\_\_\_;\qquad 10-8=\_\_,\quad 100-80=\_\_\_;\qquad \text{etc.}$$

Doing these exercises children express in words that what they do: 4 units plus 3 units is 7 units, 4 tens plus 3 tens is 7 tens, etc. This expression in words should not be taken as an explanation but it stresses the analogy of calculation with units and with tens as being decimal units of different order.

2.1. Procedures of addition and subtraction of two-digit numbers

When the addition of two two-digit numbers is represented iconically, then the elements of the arrangements representing two summands are regrouped, putting together 10-arrangements as well as the arrangements representing units. In Fig. 2 such regrouping is illustrated in the case of the sum $25 + 12$.

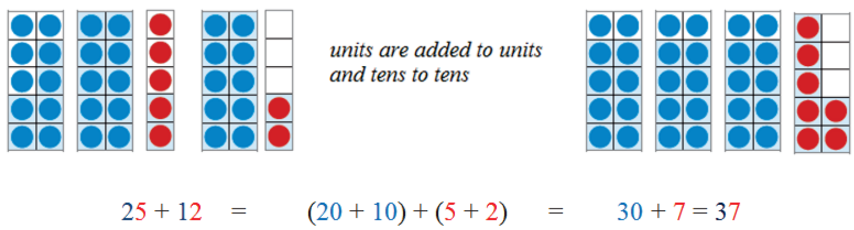


Fig. 2. Illustration of the regrouping of elements of two arrangements

The teacher is supposed to comment this procedure: “Two arrangements are seen, representing that the number 25 consists of 2 tens and 5 units and the number 12 of 1 ten and 2 units. Taken together, these arrangements represent the sum $25 + 12$. When 10-arrangements are grouped together there will be 3 of them and grouping together the units arrangements, one 7-arrangement is obtained. The new arrangement represents 37.” This comment is used to add some dynamic to the static illustrations.

On the other hand, this procedure is expressed symbolically, writing

$$15 + 12 = (20 + 10) + (5 + 2) = 30 + 7 = 37$$

In this way all intermediate steps are registered, which serves as a basis for the complete understanding. Such exercises have to be prepared in a programmed form, using place holders and leaving to children to write only the missing numbers.

Write the missing numbers

$35 + 24 = (30 + \_\_\_) + (5 + \_\_) = 50 + \_\_ = \_\_\_,$

$62 + 36 = (\_\_\_ + 30) + (\_\_ + 6) = \_\_\_ + 8 = \_\_\_,$

$52 + 27 = (\_\_\_ + \_\_) + (\_\_\_ + \_\_) = \_\_\_ + \_\_ = \_\_\_,$ *etc.*

With more place holders such exercises become somewhat more demanding, but they are easy as long as it is clear to the children what is required from them.

Subtraction of two-digit numbers also begins with the simpler cases when there is no borrowing. Such an example is illustrated in Fig. 3.

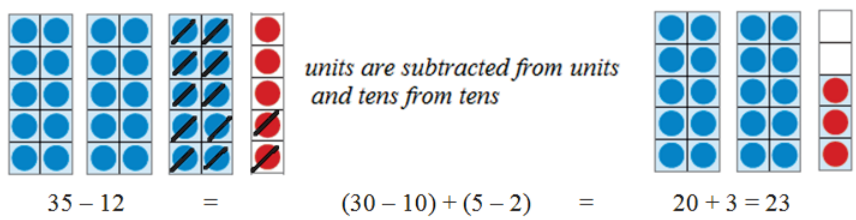


Fig. 3. Illustration showing that a sub-arrangement is taken away

The teacher comments: "We see one arrangement which consists of 3 10-arrangements and one 5 arrangement and one sub-arrangement of cancelled elements which consists of one 10-arrangement and one 2-arrangement. Taking away 2 cancelled units and 1 10-arrangement of cancelled elements, 3 units and 2 10-arrangements remain. The remaining arrangement represents 23. Hence, $35 - 12 = 23$."

Symbolic way of calculation of differences of this type should be supported by illustrations again and again in a number of cases.

Write the missing numbers

$56 - 34 = (50 - 30) + (6 - 4) = \_\_\_ + \_\_ = \_\_\_,$

$87 - 56 = (80 - \_\_\_) + (7 - \_\_) = \_\_\_ + \_\_ = \_\_\_,$

$49 - 27 = (\_\_\_ - \_\_\_) + (\_\_ - \_\_) = \_\_\_ + \_\_ = \_\_\_,$ *etc.*

2.1.1. Narrative rules. The cases of addition and subtraction that we have considered until now are performed according to the rules:

- *units are added to units and tens to tens,*
- *units are subtracted from units and tens from tens.*

Certainly, these rules direct children how to perform the operations but they say nothing how one ten is carried over or borrowed in the cases it happens. Having a latent meaning, these and similar rules have a function of suggesting how to do something and thereby they should be considered as useful for the narrative way of knowing. Much more about narrative forms can be found in the brilliant Bruner's book [2], which inspires us to think here about narrative rules in mathematics.

2.1.2. Addition of two-digit numbers when one ten is carried. Now we consider addition of 2-digit numbers when the sum of their units exceeds 9. A number of cases of such addition should be interpreted as grouping of elements of the two arrangements which represent two summands. An example of that interpretation, in the case of the sum $17 + 26$, is represented in Fig. 4.

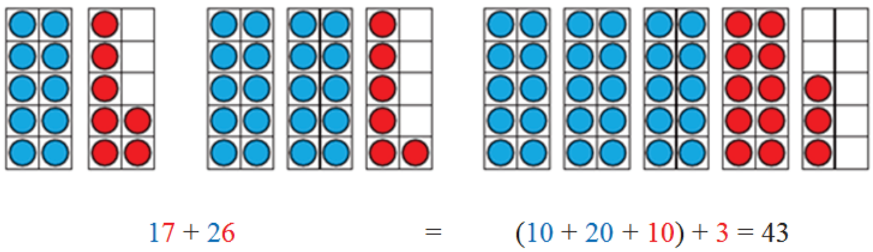


Fig. 4. Addition when one ten is carried

The teacher comments: “First two arrangements represent the sum $17 + 26$. Grouping together two red arrangements we get one red 10-arrangement and one red 3-arrangement. Altogether we have 3 blue 10-arrangements and 1 red 10-arrangement and 1 red 3-arrangement. Hence, the new arrangement represents 43. Thus, $17 + 26 = 43$. Adding units we get 3 units and 1 ten. Adding tens we get 4 tens. Altogether, it is 4 tens and 3 units.”

First exercises should follow the form of elaboration of the example in Fig. 4.

Write the missing numbers

$38 + 45 = (30 + 40 + 10) + 3 = \underline{\quad\;} + 3 = \underline{\quad\;}$,

$52 + 39 = (50 + 30 + \underline{\quad\;}) + 1 = \underline{\quad\;} + \underline{\;\;} = \underline{\quad\;}$,

$37 + 58 = (\underline{\quad\;} + \underline{\quad\;} + 10) + 5 = \underline{\quad\;} + \underline{\;\;} = \underline{\quad\;}$,

$65 + 15 = (\underline{\quad\;} + \underline{\quad\;} + \underline{\quad\;}) + \underline{\;\;} = \underline{\quad\;} + \underline{\;\;} = \underline{\quad\;}$, *etc.*

Symbolic way of expressing the addition procedure stands for the sake of clarity. For the sake of brevity, the teacher should use a picture representing addition, say that in Fig. 4, and following the way how the sum $17 + 26$ has been done, he/she says: “7 units and 6 units make 1 ten (to be carried) and 3 units. We write 3 in the place of units and we add tens: 1 ten plus 2 tens plus 1 ten (that was carried over) make 4 tens and 4 is written in the place of tens.”

To acquire good calculation skill, children are assigned a number of doing sums in this way:

$36 + 22 = \underline{\quad\;}$, $43 + 38 = \underline{\quad\;}$, $8 + 37 = \underline{\quad\;}$, $44 + 9 = \underline{\quad\;}$, etc.

2.1.3. Subtraction of two-digit numbers when one ten is borrowed. Now we consider the subtraction of two-digit numbers when the units digit of subtrahend is bigger than the corresponding digit of minuend. This procedure is illustrated in Fig. 5 in the case of the difference $43 - 17$.

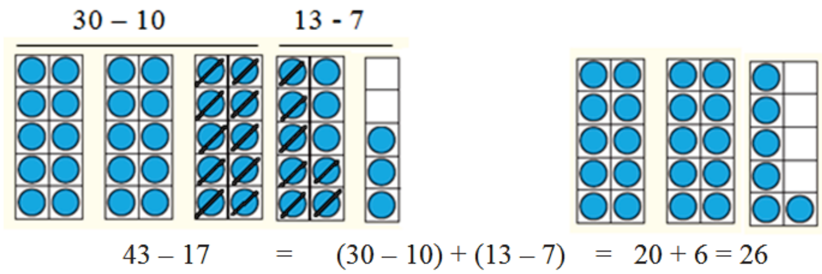


Fig. 5. Subtraction when one ten is borrowed

The teacher comments: "We see that 7 cannot be subtracted from 3. Therefore we borrow 1 ten having 13 units from which 7 units are subtracted obtaining so 6 units. Now the number of tens of the minuend is 1 less and subtracting 1 ten from 3 tens, we see that 2 tens remain."

For the sake of understanding this procedure, a number of examples (accompanied with some illustrations) should be assigned to children

Write the missing numbers

$52 - 38 = (40 - 30) + (12 - 8) = \_\_\_ + \_\_ = \_\_\_,$

$64 - 47 = (50 - \_\_\_) + (14 - \_\_) = \_\_\_ + \_\_,$

$81 - 55 = (\_\_\_ - \_\_\_) + (\_\_\_ - \_\_) = \_\_\_ + \_\_ = \_\_\_,$ *etc.*

For the sake of brevity, relying on iconic representations, say as the one in Fig. 5, the teacher explains how a difference as, for example,

$$43 - 17 = \_\_\,\_\_$$

is done: "First we try to subtract units from units but 7 cannot be subtracted from 3. Borrowing 1 ten, we subtract 7 from 13 and we write 6 in the units place. After borrowing, 3 tens remain and we subtract 1 ten from 3 tens obtaining 2 tens and writing 2 in the 10s place." In order to acquire good calculation skill, children do a number of exercises as the following ones are:

$64 - 25 = \_\_\,\_\_,\quad 71 - 49 = \_\_\,\_\_,\quad 53 - 15 = \_\_\_,\quad 60 - 31 = \_\_\_,$ etc.

2.1.4. Automatic performance of operations and the inner speech. Up to now our focus has been on understanding of calculation procedures, but now our attention will be turned to automatic performance of addition and subtraction of 2-digit numbers. We will consider how these operations are performed when two numbers are written under one another and when ones are aligned to form a column (units column) and tens are also aligned to form a column (tens column). Traditionally such operations are called vertical addition and vertical subtraction. In this case, and later in the case of arbitrary multi-digit numbers, procedures of addition and subtraction are reduced to the operating on the numbers represented by digits supposing that all entries of addition (subtraction) table are well memorized. Thus this specific case of performance of these operations deserves our full attention since in the general case, the performance of these operations runs analogously when decimal units of higher order are carried over or borrowed.

Explanation in words of addition of 2-digit numbers is rather lengthy. Let us take to consider a concrete example, say the sum of numbers 37 and 46. The operation is performed in the following three steps:

$$a)\quad \begin{array}{r} \\ 39\\ +\,46\\ \hline \\ \end{array} \qquad b)\quad \begin{array}{r} {\scriptstyle 1}\\ 39\\ +\,46\\ \hline 5\end{array} \qquad c)\quad \begin{array}{r} {\scriptstyle 1}\\ 39\\ +\,46\\ \hline 85\end{array}$$

a) Two numbers are written under one another, 9 and 6 in the units column, 3 and 4 in the 10s column.

b) When units are added to units, 1 ten and 5 units are obtained. We write 5 in the units column and 1 in the 10s column to show that 1 ten was carried over).

c) Tens are added to tens: 4 tens plus 3 tens plus 1 ten (that was carried over) make 8 tens. We write 8 in the tens column.

Similarly subtraction of 2-digit numbers is performed in three steps, shown here in the concrete case of the difference of numbers 72 and 38.

$$a)\quad \begin{array}{r} \\ 72\\ -\,38\\ \hline \\ \end{array} \qquad b)\quad \begin{array}{r} {\scriptstyle 1}\\ 72\\ -\,38\\ \hline 4\end{array} \qquad c)\quad \begin{array}{r} {\scriptstyle 1}\\ 72\\ -\,38\\ \hline 34\end{array}$$

a) The subtrahend 38 is written under the minuend 72 – 8 under 2 to form the units column and 3 under 7 to form the tens column.

b) We cannot subtract 8 from 2. Then, 1 ten is borrowed and we subtract 8 from 12, obtaining 4 and writing 4 in the units column.

c) 3 tens are subtracted from 6 tens (1 ten has been borrowed), obtaining 3 tens and writing 3 in the tens column.

It is much easier to learn by demonstration how numbers are aligned and in which places the digits have to be written. On the other hand, verbal description of these procedures becomes more and more simplified. For example, in the former case it runs as: 5 plus 8 is 15, 5 is written and 1 carried, 4 plus 3 plus 1 is 8 or similarly, and in the latter case as: 8 cannot be subtracted from 2, 1 ten is borrowed, 8 from 12 is 4 and 3 from 6 is 3 or similarly.

These simplified formulations we take as being examples of inner speech. According to L. Vygotsky [20], this kind of speech is not necessarily "speech minus sound", but a formulation which is highly elliptic, having an abbreviated syntax and strongly relying on its semantics. When externalized as oral or written speech such a formulation may not be very accurate but it functions as a "draft" which supports the activity that is carried out. In school books some printed boxes should be used to enclose such formulations. For example,

$$\begin{array}{r} 39 \\ \underline{+\,46} \end{array} \qquad \boxed{\begin{array}{l} \text{6 and 9 is 15, 5 is written,} \\ \text{1 carried, } 4+3+1=8 \end{array}}$$

With acquiring the skill, such formulations are usually forms of talking to oneself in silence.

Teachers should be aware that the inhibition of inner speech that follows calculation would have detrimental effects on automatic performance of operations.

2.2. Comments

In this section the block N_{20} has been extended up to 100, preserving permanently already established meaning of addition and using sums with known summands to produce new numbers. Thus, the set N_{100} of the first 100 natural numbers plus zero is endowed with a structure which consists of two operations—addition and subtraction together with the relation "be greater than". Symbolically, such structure is denoted by

$$\{N_{100}, +, -, <\}$$

and here the main didactical task has been purely technical—transformation of sums and differences of 2-digit numbers into their unique decimal notations.

Realization of this task is based on iconic representation of addition and subtraction (which provides meaning) and expression of the procedures of performing these operations in words and symbolically (which provides understanding). For the sake of automatic performance of these operations the formal operating on digits is performed.

Here the expression in words has the form of the teacher's comments but in the real classroom practice they should be transformed into questions accompanied with some hints. Symbolic way of expressing these procedures is primarily intended to be a ground for understanding and an example of the role of early algebra as a tool in elaborating arithmetic. Children are supposed to know to read arithmetic expressions and, understanding their structure, to complete them by writing the missing numbers. But they are not supposed to set them down except in the simplest cases of writing sums or differences.

3. Multiplication in the block of numbers up to 100

It is often said that multiplication is repeated addition i.e. those cases of addition when all summands are equal. There is nothing wrong in it, but such an approach is formal and without direct perception of those situations when we react performing operation of multiplying. However some preliminary technical activities do exist as, for example, counting in 2, 3, 4, ... , 9. Usually these are oral exercises and when counting, say in 5, the numbers 5, 10, 15, 20, ... , are recited one after the other and each of them is obtained from the preceding one adding 5 to it. In fact, it is the sequence of products that is formed in that way and such activity of counting implicitly relates multiplication and addition.

Elaboration of multiplication in the real classroom practice starts with displaying of some situations to which the products of two numbers are attached. For example,

- 1 *hand has* 5 *fingers,* 2 *hands have* $2 \cdot 5$ *fingers (and* $2 \cdot 5$ *is read as "two times five"),*
- 1 *car has* 4 *wheels,* 3 *such cars have* $3 \cdot 4$ *wheels (and* $3 \cdot 4$ *is read as "three times four"),*
- 1 *domino has* 3 *dots,* 5 *such dominos have* $5 \cdot 3$ *dots (and* $5 \cdot 3$ *is read "five times three"),* etc.

In the beginning, these and similar situations are illustrated and children are assigned to do exercises as, for example, the one in Fig. 6.

Look at the pictures and write what is required

Read what you have written.

Fig. 6. Situations to which the product of two numbers is attached

Children should not be hurried over this stage of recognition of situations to which products are attached. In a number of first lessons this process of writing products should not be followed by the question "How many are there". In that way the product is understood as a notation for a number, not a command for calculation. And traditionally, these situations were described as the cases when in each of m places there are n objects and when $m \cdot n$ denotes the total number of those objects. We use the term *multiplicative scheme* to denote such a situation and, using the language of set theory, we define it to be a family of m disjoint sets, each having n elements. Then, the product $m \cdot n$ denotes the total number of elements of the union of these sets. Of course, children develop a sense for the meaning of multiplicative scheme by experiencing a series of corresponding examples and through the activities of doing exercises. When a multiplicative scheme is perceived (or imagined) the number of sets is written as the first factor and the number of elements in each of them as the second factor. The difference in meaning between these factors contributes to the clarity of thought and should be carefully practiced in the early stage of elaboration of this theme.

Addition and multiplication should not be related formally but at the level of meaning. Namely, a multiplication scheme is also an additive scheme and the product and the sum corresponding to such a scheme can be equated as being two notations for one and the same number. Children should do a number of exercises as the following is:

Write the missing numbers

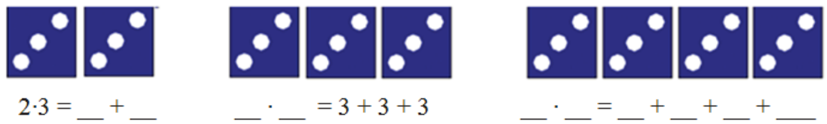


Fig. 7. Relating products and sums

Longer sums are awkward to be written as well as to be calculated. Thus, it is reasonable to confine the type of these exercises to the cases when $m = 2$, 3 or 4 (and n is a one-digit number). As a result of calculating products by calculation of the values of the related sums, a "small" multiplication table

$\cdot$	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

should be formed and its entries remembered spontaneously.

The products having a factor equal to 1 or 0 are easier to be accepted when illustrated as the cases coming at the end of some sequences of products with decreasing factors (Fig. 8).

Look at the pictures and write what is required

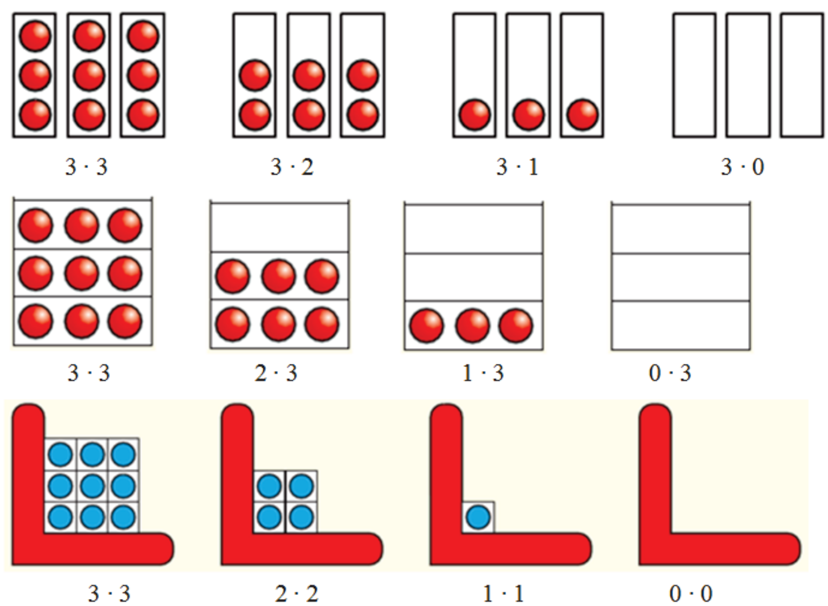


Fig. 8. Numbers 0 and 1 as factors

This is the right place for a remark that the teachers should particularly consider. Three pictures at the right end in Fig. 8—three empty frames represent the empty set. What will be representing the empty set depends on the context. For example, when boxes with marbles are under consideration, an empty box represents the empty set. This is the way how the empty set gains meaning (and to say that the empty set is the "absolute nothing", nothing of that meaning remains).

Highly regular examples of the multiplicative scheme are rectangular arrangements of object (as the one represented in Fig. 9).

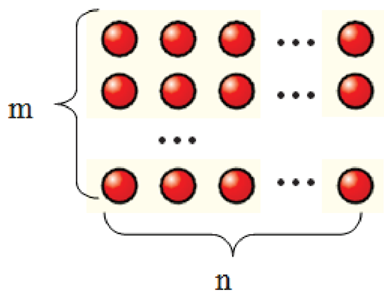


Fig. 9. A rectangular arrangement

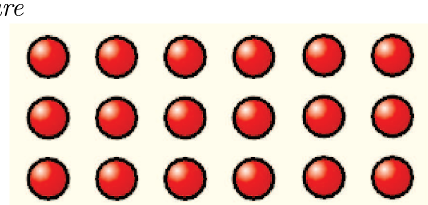
This scheme can be used to derive the *rule of interchange of factors.* Looking at the scheme:

- In each of m rows there are n elements. Altogether, it is $m \cdot n$ elements.
- In each of n columns there are m elements. Altogether, it is $n \cdot m$ elements.

Equating two expressions which denote one and the same number of elements the equality $m \cdot n = n \cdot m$ is obtained. This reasoning is based on the Cantor principle of invariance of number—the total number of elements does not depend on the ways of grouping them in rows or columns.

In the real school practice this rule should be induced doing some more concrete exercises of the following type:

Look at the picture



In each of __ (3) *rows there are* __ (6) *discs. Altogether, it is* $3 \cdot 6$ *disks.*

In each of __ (6) *columns there are* __ (3) *discs. Altogether, it is* $6 \cdot 3$ *discs.*

We write: $3 \cdot 6 = 6 \cdot 3$.

Etc.

Here the teacher emphasizes the fact that the total number of disks has been written in two ways, as $3 \cdot 6$ and $6 \cdot 3$. That is why we can equate, writing $3 \cdot 6 = 6 \cdot 3$. In this as well as in many similar cases some teacher's comments make the text of exercises clearer without complicating their own formulation. Doing a number of similar exercises children are ready to accept the rule of *interchange of factors*: *The factors may be interchanged without altering the product.*

The rule of interchange of factors should be applied to a number of cases when after applying it, the product becomes easier to be calculated.

Calculate

$$9 \cdot 3 = 3+3+3+3+3+3+3+3+3 = \underline{\qquad},$$
$$3 \cdot 9 = 9+9+9 = \underline{\qquad}.$$

What is easier to calculate, $9 \cdot 3$ *or* $3 \cdot 9$ *?*

Calculate in the easier way

$$8 \cdot 4 = 4 \cdot 8 = 8+8+8+8 = \underline{\qquad},$$
$$7 \cdot 3 = \underline{\quad} \cdot \underline{\quad} = \underline{\quad} + \underline{\quad} + \underline{\quad} = \underline{\qquad},$$
$$9 \cdot 2 = \underline{\quad} \cdot \underline{\quad} = \underline{\quad} + \underline{\quad} = \underline{\qquad}, \text{ etc.}$$

3.1. Building up the multiplication table

The block N_{100} is a natural frame within which the multiplication (division) table has to be built up. Final aim of this theme is that the children have memorized all entries into this table spontaneously, not by repeating until the entries are learnt by heart. Learning this table by rote was an exclusive practice in the traditional school. But in the contemporary school this theme is elaborated in this or that way to make the remembering easier. But it happens often that such elaboration is not a good ground for the process of remembering so that a number of children fails to learn all entries into the table by heart or if they do that temporarily, they forget them quickly. These children, when in upper classes, are often rebuked by their teachers for not knowing something elementary and important.

Here we use number pictures to make the value of some products be seen at the first glance or to be the evidence for some short and quick calculation procedures. Children learn these procedures doing a number of exercises and when a child does not know the value of a product by heart, nothing bad!, he/she is ordered to find it calculating. This is also the right place to remark that the number pictures, when once chosen, should not be changed and replaced by others. Figuratively speaking, these pictures are "iconic notations" for numbers and they play an analogous role to the decimal notations.

3.1.1. Multiplying the number 5. It is well known that the values of products having 5 for a factor are easiest to remember. The reason is evidently the fact that two fives make one ten Visually, 5-arrangements form directly number pictures, as it is seen in Fig. 10.

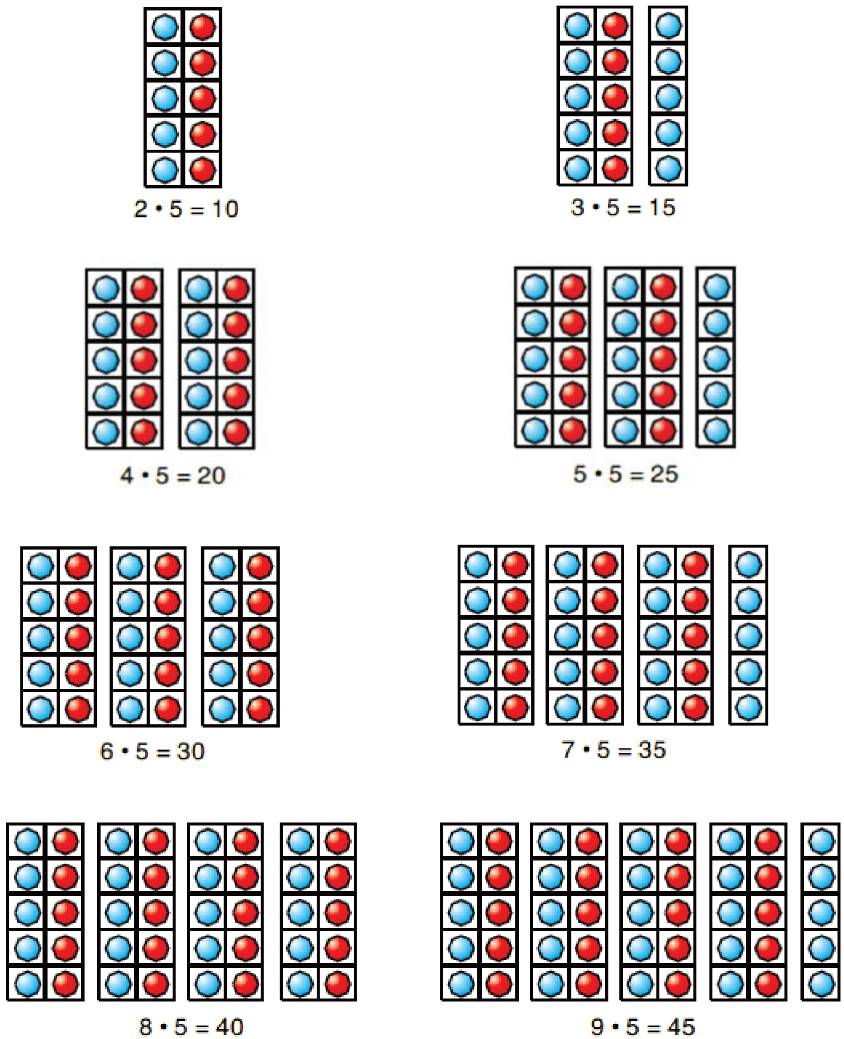
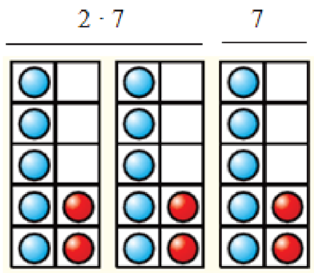


Fig.10. Multiples of five

Pointing to the pictures one by one, the teacher asks the questions: "How many fives?" and "Which number is represented?". An enlarged copy of Fig. 10 should be put on the wall of the classroom as a poster.

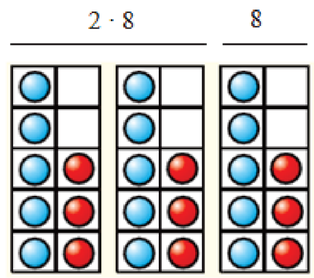
3.1.2. Products with multipliers 2 and 3. The products $2 \cdot 2$, $2 \cdot 3$, ... , $2 \cdot 9$ are the sums $2 + 2$, $3 + 3$, ... , $9 + 9$ and their values are already known from the addition table. The products $3 \cdot 3$, $3 \cdot 4$, ... , $3 \cdot 9$ are seen (Fig. 11) as the suns $2 \cdot 3 + 3$, $2 \cdot 4 + 4$, ... , $2 \cdot 9 + 9$.

Calculation of these products should be mainly be performed as the oral exercises, including also those cases obtained when the factors are interchanged.



$3 \cdot 7 = 2 \cdot 7 + 7$

$= 14 + 7 = 21$

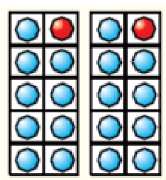


$3 \cdot 8 = 2 \cdot 8 + 8$

$= 16 + 8 = 24$

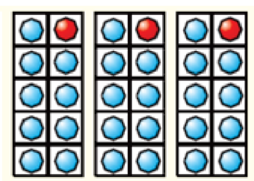
Fig. 11. Numbers 2 and 3 as multipliers

3.1.3. Multiplying the number 9. The arrangements in Fig. 11 are blue 9-arrangements with a red disk added to form 10-arrangements. Thus, it is easy to see and it is suitable to remember that $2 \cdot 9 = 20 - 2$, $3 \cdot 9 = 30 - 3$, $\ldots$, $9 \cdot 9 = 90 - 9$:



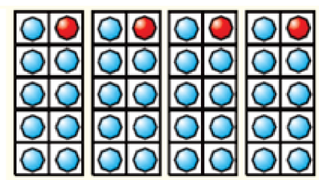
$2 \cdot 9 = 2 \cdot 10 - 2$

$2 \cdot 9 = 20 - 2 = 18$



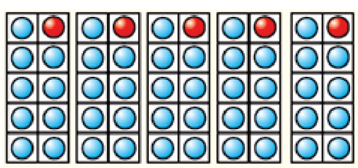
$3 \cdot 9 = 3 \cdot 10 - 3$

$3 \cdot 9 = 30 - 3 = 27$



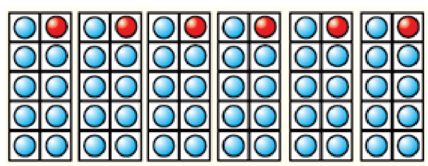
$4 \cdot 9 = 4 \cdot 10 - 4$

$4 \cdot 9 = 40 - 4 = 36$



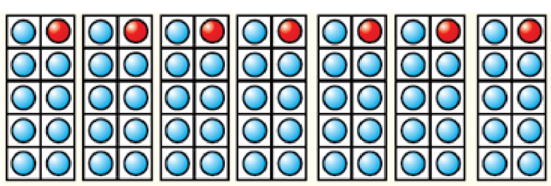
$5 \cdot 9 = 5 \cdot 10 - 5$

$5 \cdot 9 = 50 - 5 = 45$



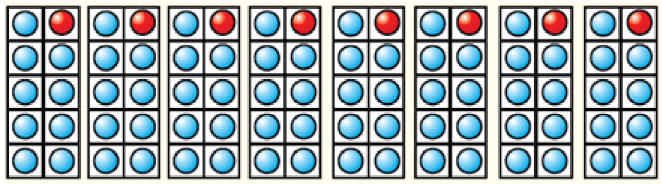
$6 \cdot 9 = 6 \cdot 10 - 6$

$6 \cdot 9 = 60 - 6 = 54$



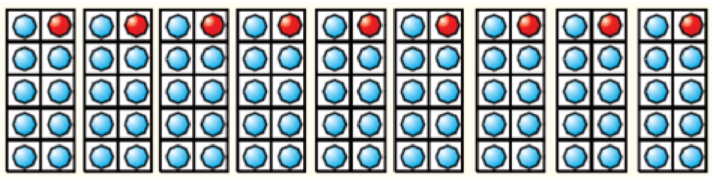
$7 \cdot 9 = 7 \cdot 10 - 7$

$7 \cdot 9 = 70 - 7 = 63$



$8\cdot 9 = 8\cdot 10 - 8$

$8\cdot 9 = 80 - 8 = 72$



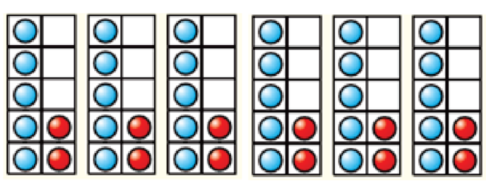
$9\cdot 9 = 9\cdot 10 - 9$

$9\cdot 9 = 90 - 9 = 81$

Fig. 12. Multiples of 9

Also as an enlarged copy, Fig. 12 should be put as a poster on the walls of the classroom.

3.1.4. Multipliers 4, 6, 7 and 8. The only products that have not been considered yet are: $4\cdot 6$, $4\cdot 7$, $4\cdot 8$, $6\cdot 6$, $6\cdot 7$, $6\cdot 8$, $7\cdot 7$, $7\cdot 8$ and $8\cdot 8$ (not counting the cases obtained by the interchange of factors). Values of these products are also somewhat more difficult to be remembered, but the calculation in all these cases goes in one and the same way. This way consists of splitting the products into the sum of two easier ones—one having a factor equal to 5 (see 3.1.1) and the other one having a factor equal to 2 or 3 (see 3.1.2). Two examples illustrated in Fig. 13 suggest clearly this procedure.



$6\cdot 7 = 6\cdot 5 + 6\cdot 2$

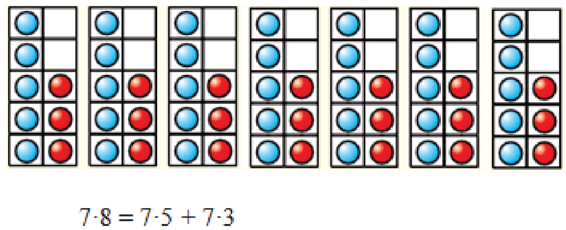


Fig. 13. Splitting products into two easier ones

This procedure can be somewhat schematized and instead of writing

$$6 \cdot 7 = 6 \cdot 5 + 6 \cdot 2 = 30 + 12 + 42, \quad 7 \cdot 8 = 7 \cdot 5 + 7 \cdot 3 = 35 + 21 = 56, \text{ etc.}$$

children would write as follows

$$6 \cdot \underset{5\ \ 2}{\underset{\bigwedge}{7}} = 30 + 12 = 42, \qquad 7 \cdot \underset{5\ \ 3}{\underset{\bigwedge}{8}} = 35 + 21 = 56, \text{ etc.}$$

Children should do a larger number of exercises of this type in order to form a good calculation skill.

3.2. Various aspects of the multiplicative scheme

In the real classroom practice the term multiplicative scheme is not used. Instead of it, a long series of examples is given to children and the tasks are formulated in such a way that they put forward the activity of multiplying (or dividing). Not all examples are formulated in the way that the idea of a family of disjoint equipotent sets is directly suggested. For example when we say that on 3 hands there are $3 \cdot 5$ fingers, we do not certainly mean that a hand is the set of its fingers. In fact the fingers on a hand form a set. Similarly some ambiguity appears when higher and lower units of measure are compared. For example, we say that 3 meters contain $3 \cdot 10$ decimeters. Taken as a line segment, the meter can be seen divided into 10 smaller segments placed end to end and each representing a decimeter. In this case the idea of a set is associated with the set of these smaller segments. Therefore, some situations are not clear enough to be recognized immediately as a model of the multiplicative scheme, though such a model can be discerned in all of them. When necessary, the teacher should represent (or describe) such situations in a more distinct way, inducing so his/her pupils' understanding.

We have already defined the multiplicative scheme to be a family of m disjoint sets each having n elements. When the numbers m and n are given and the total number p of elements of these sets is to be found, then we speak of a multiplication task following a multiplicative scheme. When the number p and one of the numbers m or n is given and the other one is to be found, then we speak of the division task following a multiplicative scheme.

As a nice model for the multiplicative scheme we can take m boxes, each containing n marbles. There is an asymmetry between the meaning of m denoting the number of sets (boxes) and n denoting the number of elements (marbles). A scheme shown in Fig. 9 is much more symmetric and can be seen in the same time as m rows each of which has n elements (discs) or as n columns each of which has m elements (discs).

Given a multiplicative scheme we can put its sets into a sequence (A_i), $i = 1, 2, \ldots, m$ and also the elements of each A_i into a sequence (a_{ij}), $j = 1, 2, \ldots, n$. Thus, to each element a_{ij} the pair of numbers (i, j) corresponds and this correspondence is 1–1 and onto. The set of all pairs (i, j) is the direct product $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ and thereby $m \cdot n$ can be taken as the number of elements of this product of sets. When A and B are arbitrary sets having m and n elements respectively, then their direct product $A \times B$ has $m \cdot n$ elements. As we see it, this product can also be taken for the multiplicative scheme. To the abstract concept of the direct product of sets, a rectangular arrangement as an iconic sign corresponds. Pairs of numbers indicating the place of an element in a row and in a column constitute a set which is the direct product of the set $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$.

Let us note that, in set theory, when m and n are arbitrary cardinal numbers, $m = \mathrm{card}(A)$, $n = \mathrm{card}(B)$ the product $m \cdot n$ is defined to be the cardinal number of the set $A \times B$, i.e. $m \cdot n = \mathrm{card}(A \times B)$.

3.3. Further properties of multiplication

The terms "sum" and "product" have two different meanings—the syntactic one, when they are understood as the "pieces of writing" and the semantic one, when they are understood as numbers denoted by such expressions. The number denoted by a product (sum) is called the *value of that product (sum)*. But the phrase "*find the value of a product (sum)*" is understood as a command to transform a given product (sum) so that its decimal notation is obtained. Properties of multiplication are used to carry out such a transformation. When these properties are expressed symbolically, the brackets are used as a syntactic sign and, dealing with the properties of addition, children have already learnt their role as a command "do first what has been written in brackets".

In order to experience some situations where the brackets are used, children do exercises as the following one:

Fill in the missing numbers to show that

(i) *the given products are multiplied by* 7

$$\begin{array}{ccc} a)\ \ 3 \cdot 5 & b)\ \ 6 \cdot 2 & c)\ \ 4 \cdot 3 \\ 7 \cdot (\_\_ \cdot \_\_) & \_\_ \cdot (6 \cdot 2) & \_\_ \cdot (\_\_ \cdot \_\_), \text{etc.} \end{array}$$

(ii) *the given numbers are multiplied by* $3 \cdot 8$

$$\begin{array}{ccc} a)\ \ 4 & b)\ \ 3 & c)\ \ 2 \\ (3 \cdot 8) \cdot \_\_ & (\_\_ \cdot \_\_) \cdot 3 & (\_\_ \cdot \_\_) \cdot \_\_,\ \text{etc.} \end{array}$$

To emphasize the role of brackets as a command for calculation, formal exercises should be assigned to children as the following one is:

Calculate

$$2 \cdot (6 \cdot 5) = 2 \cdot \_\_\_ = \_\_\_, \quad (6 \cdot 5) \cdot 2 = \_\_\_ \cdot 2 = 2 \cdot \_\_\_ = \_\_\_,$$
$$6 \cdot (5 \cdot 2) = 6 \cdot \_\_\_ = \_\_\_, \quad (5 \cdot 2) \cdot 6 = \_\_\_ \cdot 6 = 6 \cdot \_\_\_ = \_\_\_, \text{ etc.}$$

To experience further those cases when brackets are used, children have also to do a number of exercises as this one:

Fill in the missing numbers to show that

(i) *the given products are multiplied by* 7*:*

a) $3 \cdot 5$ *b*) $5 \cdot 2$ *c*) $4 \cdot 3$

$7 \cdot (\_\_ \cdot \_\_)$ $\_\_ \cdot (5 \cdot 2)$ $\_\_ \cdot (\_\_ \cdot \_\_)$.

(ii) *the given numbers are multiplied by* $3 \cdot 8$*:*

a) 4   *b*) 3   *c*) 2

$(3 \cdot 8) \cdot \_\_$ $(\_\_ \cdot \_\_) \cdot 3$ $(\_\_ \cdot \_\_) \cdot \_\_$, etc.

3.3.1. The rule of association of factors. All rules of arithmetic should be interpreted intuitively and treated as acceptable principles—not as the provable facts. Let us consider an example illustrating such a situation.

Look at the picture

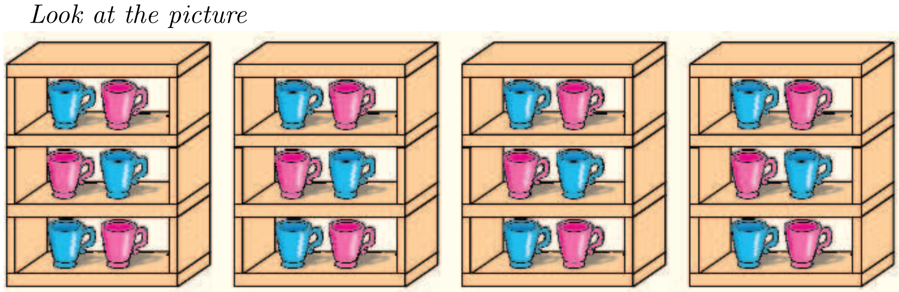


Fig. 14

You see 4 *cupboards, each having* 3 *shelves and on each shelf* 2 *cups.*

The number of shelves is $\_\_ \cdot \_\_$ *and the number of cups is* $(4 \cdot 3) \cdot \_\_$.

In each cupboard there are $\_\_ \cdot \_\_$ *cups. Altogether, there are* $\_\_ \cdot (3 \cdot 2)$ *cups.*

The total number of cups we have denoted—once as $(4 \cdot 3) \cdot 2$ *and the other time as* $4 \cdot (3 \cdot 2)$*. Both* $(4 \cdot 3) \cdot 2$ *and* $4 \cdot (3 \cdot 2)$ *denote one and the same number and we can write* $(4 \cdot 3) \cdot 2 = 4 \cdot (3 \cdot 2)$*.*

To demonstrate that this equality holds true when the numbers 4, 3 and 2 are replaced by any three other numbers, the teacher continues to elaborate further the above example.

What would you write if there were

a) 6 *cupboards*	*b*) 7 *cupboards*	*c*) 9 *cupboards*
$(\_\_ \cdot 3) \cdot 2 = 6 \cdot (\_\_ \cdot \_\_)$	$(7 \cdot \_\_) \cdot \_\_ = 7 \cdot (\_\_ \cdot \_\_)$	$(\_\_ \cdot \_\_) \cdot \_\_ = \_\_ \cdot (\_\_ \cdot \_\_)$

What would you write if there were

a) 4 *shelves*	*b*); 6 *shelves*	*c*) 8 *shelves*
$(4 \cdot \_\_) \cdot 2 = \_\_ \cdot (4 \cdot \_\_)$	$(\_\_ \cdot 6) \cdot 2 = \_\_ \cdot (6 \cdot \_\_)$	$(\_\_ \cdot \_\_) \cdot \_\_ = \_\_ \cdot (\_\_ \cdot \_\_)$

a) 3 *cups*	*b*) 4 *cups*	*c*) 7 *cups*
$(\_\_ \cdot 3) \cdot 3 = \_\_ \cdot (3) \cdot \_\_$	$(\_\_ \cdot \_\_) \cdot 4 = \_\_ \cdot (\_\_ \cdot 4)$	$(\_\_ \cdot \_\_) \cdot \_\_ = \_\_ \cdot (\_\_ \cdot \_\_)$

What would you write if there were 3 *cupboards each having* 6 *shelves with* 4 *cups on each shelf?*

$$(\_\_ \cdot \_\_) \cdot \_\_ = \_\_ \cdot (\_\_ \cdot \_\_)$$

On the basis of similar examples children are induced to accept the *rule of association of factors: In whatever way the factors are associated the value of the product stays unchanged.*

3.3.1.1. Invariant forms of arithmetic rules. In the previous exercises we have established the equality $(4 \cdot 3) \cdot 2 = 4 \cdot (3 \cdot 2)$ and we have seen that, when the triple of numbers 4, 3, 2 is replaced by any other triple, say 3, 6, 4 the equality $(3{\cdot}6){\cdot}4 = 3{\cdot}(6{\cdot}4)$ holds to be true again. For such a form of an arithmetic equality we will say that it has the *invariant form*. The equalities, as for example, $3+5 = 5+3$, $7+(10+6) = (7+10)+6$, $3 \cdot 7 = 7 \cdot 3$, $(4 \cdot 6) \cdot 3 = 4 \cdot (6 \cdot 3)$ all have the invariant form. Adding some restrictions which ensure that subtrahends are not larger than minuends, the equalities $(6+8)-4 = 6+(8-4)$, $10-(6+4) = (10-6)-4$ also have the invariant form. The equalities, as for example, $7+16 = 17+6$, $(7+16)+3 = 17+(6+3)$, $14-4 = 6+4$, $10-9 = 5-4$, etc. are all true, but they do not have the invariant form. The form $10-9 = 5-4$ is not invariant, but $10-9 = (10-5)-(9-5)$ is invariant what is easy to see when the triple 10, 9, 5 is replaced by any other triple k, l, m.

3.3.1.2. Procedural, rhetoric and symbolic way of expressing rules of arithmetic. The way of expressing a rule of arithmetic using particular numbers and in invariant form we call *procedural*. In the time before the creation of symbolic algebra, the procedures of solving algebraic equations would be demonstrated in a particular case and it was expected that such a procedure would work equally in all other similar cases.

We call *rhetoric* the way of expressing a rule of arithmetic in words. And when letters are used to denote arbitrary natural numbers, we call *symbolic* the way of expressing a rule in the form of a literal equation.

To establish the rule of association of factors in symbolic form, we use the model of k packages, each containing m boxes and each box containing n marbles. Then, the number of boxes is $k \cdot m$. The total number of marbles is $(k \cdot m) \cdot n$. In m boxes there are $m \cdot n$ marbles. The total number of marbles is $k \cdot (m \cdot n)$. Equating two expressions denoting one and the same number of marbles, we get $(k \cdot m) \cdot n = k \cdot (m \cdot n)$.

The model of packages of boxes with marbles can be taken as an example of the *multiplicative scheme for the triple product* which has a general meaning. Expressed in general terms, such a scheme can be defined to be *collection of k families of sets, each family having m sets and each set having n elements.*

The way how the rules are applied, though it can seem obvious to grown-ups, is something what has to be demonstrated to children in a suitable way. To do it by doing some specially prepared example is probably the best way.

Think of the

(1) *rule of the interchange of factors*

(2) *rule of association of factors.*

When you apply rule (1)

to	*you get*	*and*	*you can write*
$3 \cdot 8$	$8 \cdot 3$		$3 \cdot 8 = 8 \cdot 3$
$7 \cdot 5$	$\_\_ \cdot \_\_$		$\_\_ \cdot \_\_ = 5 \cdot 7$
$4 \cdot (2 \cdot 5)$	$\_\_ \cdot (5 \cdot 2)$		$4 \cdot (2 \cdot 5) = \_\_ \cdot (\_\_ \cdot \_\_)$
$(3 \cdot 8) \cdot 4$	$(\_\_ \cdot \_\_) \cdot 4$		$(3 \cdot 8) \cdot 4 = (\_\_ \cdot \_\_) \cdot \_\_$, etc.

When you apply rule (2)

to	*you get*	*and*	*you can write*
$(3 \cdot 5) \cdot 4$	$3 \cdot (5 \cdot 4)$		$(3 \cdot 5) \cdot 4 = \_\_ \cdot (\_\_ \cdot \_\_)$
$8 \cdot (5 \cdot 4)$	$(8 \cdot 5) \cdot 4$		$8 \cdot (5 \cdot 4) = (\_\_ \cdot \_\_) \cdot \_\_$, etc.

Write in ( ) 1 *or* 2 *to indicate which rule has been applied:*

( )	( )	( )	( )
$4 \cdot 6 = 6 \cdot 4$,	$3 \cdot (7 \cdot 2) = (3 \cdot 7) \cdot 2$,	$3 \cdot (7 \cdot 2) = 3 \cdot (2 \cdot 7)$,	$(3 \cdot 7) \cdot 2 = 3 \cdot (7 \cdot 2)$,

etc.

A particularly important exercise is the following one:

Think of the

(1) *rule of the interchange of factors*

(2) *rule of association of factors*

and write 1 *or* 2 *in* ( ) *to indicate which of these rules has been applied:*

$$
\begin{aligned}
& \quad\ (\ \ ) \qquad\quad (\ \ ) \qquad\quad (\ \ ) \qquad\quad (\ \ ) \\
3\cdot(4\cdot5) &= 3\cdot(5\cdot4) = (3\cdot5)\cdot4 = (5\cdot3)\cdot4 = 5\cdot(3\cdot4) \\
& \quad\ (\ \ ) \qquad\quad (\ \ ) \qquad\quad (\ \ ) \qquad\quad (\ \ ) \\
&= 5\cdot(4\cdot3) = (5\cdot4)\cdot3 = (4\cdot5)\cdot3 = 4\cdot(5\cdot3) \\
& \quad\ (\ \ ) \qquad\quad (\ \ ) \qquad\quad (\ \ ) \\
&= 4\cdot(3\cdot5) = (4\cdot3)\cdot5 = (3\cdot4)\cdot5.
\end{aligned}
$$

You see that the factors of a triple product can be associated in both ways and in an arbitrary order.

This is the case of "derivation" of the generalized association rule. Although, let us be reserved to speak of derivation of rules when a ground for it is not set clearly and when deductive thinking is not characteristic for the child of this age.

3.3.2. The rules of multiplication of sums and differences. In the classroom practice, rectangular arrangements are particularly suitable form of the multiplicative scheme used for establishing the rules of multiplication of sums and differences.

Look at the arrangements

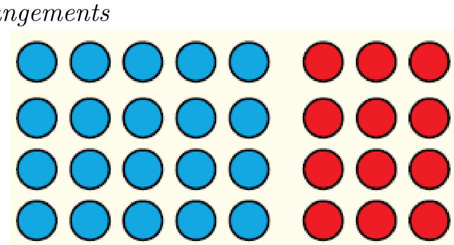


Fig. 15

There are 5 *blue and* 3 *red disks in each row. It is* $5+3$ *disks. In* 4 *rows there are* $4\cdot(5+3)$ *disks. In* 4 *rows there are* $4\cdot5$ *blue disks and* $4\cdot3$ *red discs. Altogether, it is* $4\cdot5+4\cdot3$ *discs. In one case the total number of disks is written as* $4\cdot(5+3)$ *and in the other as* $4\cdot5+4\cdot3$. *Equating we get*

$$4\cdot(5+3)=4\cdot5+4\cdot3.$$

Varying the number of blue and red discs as well as the number of rows, the teacher induces his/her pupils to accept the equalities obtained when the triple 4, 5, 3 is replaced by any other triple of numbers (from N_{100}). (See also elaboration of examples in 3.3.1).

Using the same arrangement (Fig. 15) the number of blue discs is written in two ways as $4\cdot(8-3)$ and as $4\cdot8-4\cdot3$. Equating the equality

$$4\cdot(8-3)=4\cdot8-4\cdot3$$

is established.

Expressed in words, these rules read:

Multiplying a sum two summands are multiplied and those products are added together.

Multiplying a difference the minuend and the subtrahend are multiplied and from the former product the latter is subtracted.

A very precise verbal formulation of these rules would be a burden on children. These narrative forms suggest the corresponding procedures clearly.

Now let us consider the model consisted of k boxes, each containing m blue and n red marbles. Then, the total number of marbles can be expressed in two ways, as $k \cdot (m+n)$ or as $k \cdot m + k \cdot n$. Thus, the general equality $k \cdot (m+n) = k \cdot m + k \cdot n$ is obtained.

If in each of k boxes there are m marbles of which n are red and the rest are blue. Then, the total number of blue marbles can be written as $k \cdot (m-n)$ or as $k \cdot m - k \cdot n$. Equating these expressions, the equality $k \cdot (m-n) = k \cdot m - k \cdot n$ is obtained.

3.4. Comments

Some rules of arithmetic are the field axioms as, for example, commutative and associative laws for addition and multiplication, distributive law, etc. These axioms express fundamental properties of addition and multiplication and they can be used to derive all other properties of these operations. On the other hand, in didactics of mathematics, the properties of operations are called rules of arithmetic and each of them is considered to be independent of the others (even in the case when a property is easily derived from the others). This is also the right place to notice that all these rules have been established when expressions related to the idea of different grouping of elements of a set were equated. Therefore, we see here Cantor principle of invariance of number at work.

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