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A SOMEWHAT UNEXPECTED CONCAVITY

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Abstract. This classroom note considers the slightly counterintuitive concavity of a rational function.

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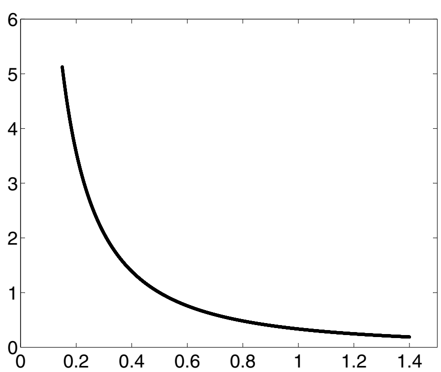
Introduction

As many calculus students will recognize, $1/x$ for $x > 0$ is a convex function, and so is $1/x^2$, or any inverse power of x. But what about, e.g., $1/(x+x^2)$? Based on a small nonscientific experiment in my classes, most of the students seemed to think that this function was also convex, and they were right—it is (see Figure 1 on the left). However, their intuition broke down with $1/(x+x^{20})$, which they also thought was convex. They were wrong! There is a finite interval of the positive real axis on which that function is concave (see Figure 1 on the right).



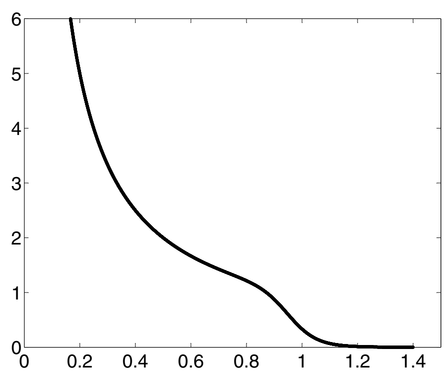


Fig. 1. The functions $1/(x+x^2)$ (left) and $1/(x+x^{20})$ (right)

The reason for this phenomenon is the rapid change in the function's leading behavior when x crosses the value 1. When $x < 1$, x dominates $x + x^{20}$, so that the function essentially behaves like $1/x$, whereas for $x > 1$, x^{20} dominates and the functions's behavior is essentially that of $1/x^{20}$. This means that, as x crosses 1 from left to right, the function drops down abruptly to a much lower value. Its derivative must therefore decrease precipitously (in this case, become more negative), making the second derivative negative, thereby causing the function's shape to become concave. As x continues to increase, the function once again becomes convex.

Clearly, the behavior of a function of the form $1/(x^p + x^q)$ with $p, q > 0$ for $x > 0$ depends on the relative magnitudes of p and q. In what follows, we will investigate the slightly more general function

$$f(x) = \frac{1}{x^p + \alpha x^q}$$

for $x > 0$, with $p, q, \alpha > 0$ and $q > p$. It is an easy calculus exercise to compute the interval on which f is concave (if such an interval exists). Let us consider the more interesting problem of deriving a condition on the parameters of f that guarantees concavity. We begin by computing the derivatives of f:

$$\begin{aligned}
f'(x) &= -\frac{px^{p-1} + \alpha q x^{q-1}}{(x^p + \alpha x^q)^2},\\
f''(x) &= \frac{2(px^{p-1} + \alpha q x^{q-1})^2 - (p(p-1)x^{p-2} + \alpha q(q-1)x^{q-2})(x^p + \alpha x^q)}{(x^p + \alpha x^q)^3}\\
&= \frac{\big(\alpha^2 q(q+1)x^{2(q-p)} - \alpha(2(q-p)^2 - q(q+1) - p(p+1))x^{q-p} + p(p+1)\big)x^{2(p-1)}}{(x^p + \alpha x^q)^3}.
\end{aligned}$$

This means that the sign of f'', which determines the shape of f, is determined by the first factor in the numerator of f''. Setting $y = x^{q-p}$, this factor becomes

$$g(y) = \alpha^2 q(q+1)y^2 - \alpha(2(q-p)^2 - q(q+1) - p(p+1))y + p(p+1).$$

If the quadratic g has two positive roots y_1 and y_2 with $y_1 < y_2$, then $g(y) < 0$ for any y such that $y_1 < y < y_2$, implying that $f''(x) < 0$ for any x such that $y_1^{1/(q-p)} < x < y_2^{1/(q-p)}$. Since its leading and constant coefficients are positive, the quadratic g can have two positive or two negative roots, the latter being of no interest to us. The roots will be positive if and only if both of the following inequalities are satisfied:

$$2(q-p)^2 > q(q+1) + p(p+1)$$

and

$$(2(q-p)^2 - q(q+1) - p(p+1))^2 > 4q(q+1)p(p+1).$$

This is equivalent to requiring that

$$2(q-p)^2 > q(q+1) + p(p+1) + 2\sqrt{q(q+1)p(p+1)},$$

which can be rewritten as

$$2(q-p)^2 > \big(\sqrt{q(q+1)} + \sqrt{p(p+1)}\big)^2,$$

i.e.,

$$\sqrt{2}q - \sqrt{q(q+1)} > \sqrt{2}p + \sqrt{p(p+1)}. \tag{1}$$

Inequality (1) is therefore a necessary and sufficient condition for the function f to become concave on an interval. It depends on the powers p and q, but not on the parameter α. The interval on which f is concave is $(y_1^{1/(q-p)}, y_2^{1/(q-p)})$, where y_1 and y_2 are the roots of g (which do depend on α).

We conclude by examining inequality (1) a little further. First of all, $p > 0$, which means that $\sqrt{2}q - \sqrt{q(q+1)} > 0$, and therefore $q > 1$. Let us now compute the threshold value of q that makes f concave on an interval, as a function of p. With $\gamma = \sqrt{2}p + \sqrt{p(p+1)}$, inequality (1) becomes

$$\sqrt{2}q - \gamma > \sqrt{q(q+1)},$$

which, since necessarily $q \geq \gamma/\sqrt{2}$, is equivalent to

$$2q^2 - 2\sqrt{2}\gamma q + \gamma^2 > q^2 + q,$$

i.e.,

$$q^2 - (1 + 2\sqrt{2}\gamma)q + \gamma^2 > 0. \tag{2}$$

The left-hand side of inequality (2) is is a quadratic in q, which is positive outside the interval determined by its positive roots. The reader is invited to prove that the smaller of these roots is less than $\gamma/\sqrt{2}$, which means that inequality (2) implies that

$$q > \frac{1}{2}\left(1 + 2\sqrt{2}\gamma + \sqrt{4\gamma^2 + 4\sqrt{2}\gamma + 1}\right). \tag{3}$$

The right-hand side of inequality (3) represents the threshold value of q causing f to be concave on an interval. Figure 2 shows this threshold value as a function of p for small values of p on the left and for larger values of p on the right.

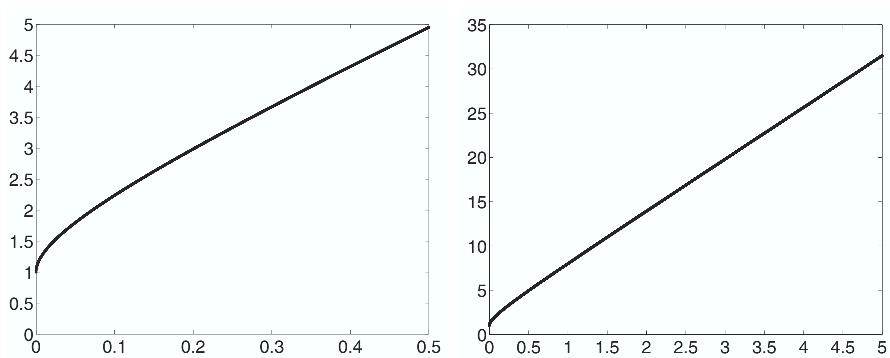


Fig. 2. Threshold value for q as a function of p

A suggested exercise for the reader is to derive directly from inequality (1) the following simple sufficient condition for f to be concave on an interval p:

$$q \geq \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)p + \frac{1}{\sqrt{2}-1}.$$

When $p = 1$, as in the example we started out with, this condition becomes $q \geq 8.2427$ which is clearly satisfied for $q = 20$. For comparison, the exact condition from (3) is $q > 8$.

Although it would certainly be more complicated than this note, it may be an interesting project to investigate what happens when the function f has additional powers of x in its denominator.

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