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MONOTONICITY OF CERTAIN RIEMANN-TYPE SUMS

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There is frequently more to be learned from the unexpected questions of a child than the discourses of men.
John Locke

We hear only those questions for which we are in a position to find answers.
Friedrich Nietzsche

Abstract. In this short note we prove with elementary techniques that the sequence $x_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$ is increasing and its limit is $\frac{\pi}{4}$. Moreover, we give a sufficient condition for the monotonicity of some Riemann-type sums assigned to uniform subdivisions as a function of the number of the intervals from the subdivision. This mathematical content came up in a group discussion during an IBL centered teacher training activity and reflects a crucial problem is implementing IBL teaching attitudes in the framework of a highly scientific curricula (such as the Romanian mathematics curricula for upper secondary school).

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1. Introduction

The main aim of this note is to reflect in a concrete mathematical context on the two statements from the motto. The background and the necessity of such a reflection is the new European Inquiry Based Learning (IBL) trend in teaching of mathematics. In the last decade it became clear that Europe needs more scientists (see [2]) and that in order to attain this goal the renewal of mathematics and science education is needed (see [4]). The Rocard-report [4] recommends a more extensive use of inquiry based learning. Since then several European projects (at local and international level) focused on IBL have been started (for a brief overview see the project coordinators network at `http://proconet.ph-freiburg.de/`). The basic ideas of IBL goes back to the work of John Dewey, but nowadays IBL is used for a large variety of pedagogies that allow students to construct their own knowledge based on inquiries. The use of students centered pedagogies needs a different student-teacher communication, different attitudes both from students and teachers and well designed, rich problem situations in which the inquiry can be done.

However there are strong evidence on the efficiency of this method (see [5]), the implementation process needs a lot of care and effort. According to the philosophy of inquiry based learning the main focus of teaching is not only the content itself, but the way knowledge is constructed by students, the learning process which takes place in well chosen contexts. For many teachers, who have been using mostly frontal teaching methods (with a long history of success) this philosophy is completely new. During an IBL activity students are encouraged to ask questions. But in the upper secondary level these questions can lead to very deep problems that teachers are not prepared to handle. In the traditional setting many natural questions arose without teachers being able to give answers on the existing knowledge level of the students (this problem is a consequence of the curricula construction and not the problem of teachers training), but most of these questions were not heard (as in the motto) or only a superficial answer was given to them (which postpones the answer until forgetting the question). In an IBL setting this attitude needs to be revised and changed. In this short note we give an answer to a question raised in a group discussion during an IBL centered teacher training activity and reflects a crucial problem is implementing IBL teaching attitude. This problem is also deeply rooted in the history of mathematical ideas, mainly the problem of measuring. The training session was focused on the introduction of the Riemann integral based on some real world problems which led to the approximation of the area of a planar domain bounded by the coordinate axes, the graph of the function $f : [0,1] \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$ and the line $x = 1$. The idea was to use students' former knowledge namely concepts and properties related to sequences and approximation. In this framework the following problems arose:

PROBLEM 1. Prove that the sequence $(x_n)_{n\geq 1}$ with general term

$$x_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2}, \quad n \geq 1,$$

is increasing. (This problem can also be found in [1].)

PROBLEM 2. Prove that the sequence $(y_n)_{n\geq 1}$ with general term

$$y_n = \sum_{k=0}^{n-1} \frac{n}{n^2+k^2}, \quad n \geq 1,$$

is decreasing.

PROBLEM 3. Find the limit of the sequence $(x_n)_{n\geq 1}$, defined by

$$x_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2}, \quad n \geq 1$$

using elementary methods (without derivatives or integral calculus).

REMARK 1. The terms of these two sequences ($(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$) are lower and upper bounds for the area in question, while the limit could be interpreted as the area of the domain.

2. Proofs and strategies

Solution of Problem 1. The general term of the sequence $(x_n)_{n\geq 1}$ can be written as

$$x_n = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2},$$

so we have to study the monotonicity of the sum

$$x_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right),$$

where $f : [0,1] \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$. This is the sum of areas of the rectangles constructed on the intervals $\left[\frac{k-1}{n}\frac{k}{n}\right]$ with height $f\left(\frac{k}{n}\right)$ for all $1 \leq k \leq n$. Using Figure 1, we have to compare the area of two systems of rectangles. These systems differ in the rectangles marked with $+$ and $-$, so it is sufficient to prove that each rectangle marked with $+$ has greater area than the right neighbour rectangle marked with $-$. This inequality can be expressed as

$$(1)\quad \left(\frac{k}{n+1}-\frac{k-1}{n}\right)\left(f\left(\frac{k}{n+1}\right)-f\left(\frac{k}{n}\right)\right) \geq \left(\frac{k}{n}-\frac{k}{n+1}\right)\left(f\left(\frac{k}{n}\right)-f\left(\frac{k+1}{n+1}\right)\right),$$

where $1 \leq k \leq n-1$. In an equivalent form we have

$$f\left(\frac{k}{n}\right) \leq \frac{n+1-k}{n+1} f\left(\frac{k}{n+1}\right) + \frac{k}{n+1} f\left(\frac{k+1}{n+1}\right).$$

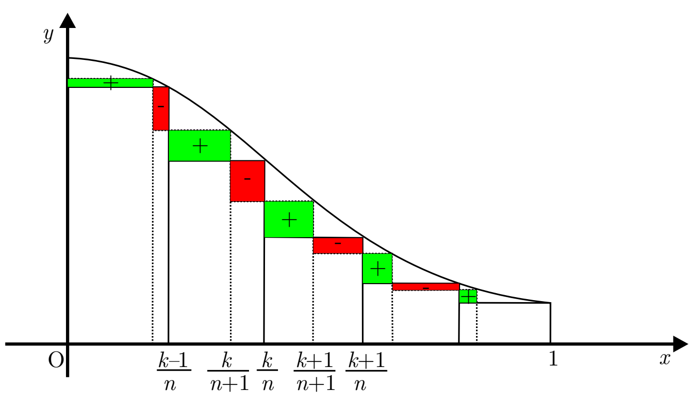

Fig. 1. Subdivisions for n and $n+1$ for the function $f(x) = \frac{1}{1+x^2}$.

Using the explicit form of the function we can rewrite the previous inequality in the following equivalent forms:

$$\frac{(n+1)(n+1-k)}{(n+1)^2+k^2} + \frac{(n+1)k}{(n+1)^2+(k+1)^2} \geq \frac{n^2}{n^2+k^2},$$

$$\left((n+1)^4+(n+1)^2(k+1)^2-k(2k+1)(n+1)\right)\left(n^2+k^2\right)$$
$$\geq n^2\left((n+1)^2+k^2\right)\left((n+1)^2+(k+1)^2\right),$$
$$(n+1)^4k-n^2(n+1)(2k+1)+(2n+1)k(k+1)^2-k^2(2k+1)(n+1)\geq 0,$$
$$n^4k+(2k-1)n^3+(4k-1)n^2+n(3k^2+6k)+2k-k^3+k^2\geq 0$$

Throughout all these inequalities $1\leq k\leq n-1$. The last inequality is true for $k\geq 1$ because $3k^2n\geq k^3$ and the rest of the terms are all positive. Hence (1) is true and this guaranties that $(x_n)_{n\geq 1}$ is increasing. ∎

Remark 2. Problem 2 can be solved using a similar reasoning by comparing areas of rectangles. This is left to the reader as a good exercise.

Remark 3. It is easy to see that $\frac{1}{2}\leq x_n\leq 1$, hence the sequence is also bounded, and this implies the convergence of the sequence. Using the definition of the Riemann integral we could easily find the limit of the sequence without the study of convergence:

$$\lim_{n\to\infty} x_n=\int_0^1\frac{dx}{1+x^2}=\frac{\pi}{4}.$$

Solution of Problem 3. First we observe that

$$\operatorname{arctg}\frac{k+1}{n}-\operatorname{arctg}\frac{k}{n}=\operatorname{arctg}\frac{n}{n^2+k^2+k},$$

hence there is a telescopic expansion for the sum $\sum_{k=1}^{n}\operatorname{arctg}\frac{n}{n^2+k^2+k}$. Using this we need to establish inequalities between $\operatorname{arctg}\frac{n}{n^2+k^2+k}$ and $\frac{n}{n^2+k^2}$. For this we recall the following basic trigonometric inequalities and their geometric proof:

$$x-\frac{x^3}{2}<\operatorname{arctg}x<x,\qquad \text{if } x\in[0,\infty). \tag{2}$$

Remark 4. In the first expression of the previous relations $\frac{x^3}{2}$ can be replaced with $\frac{x^3}{3}$.

Consider the trigonometric circle as in Figure 2. The area of the circular sector AOC is $\frac{t}{2}$, the area of the triangle AOB is $\frac{\operatorname{tg}t}{2}$, hence $t<\operatorname{tg}t$ and this implies $\operatorname{arctg}x<x$. On the other hand, the area of triangle AOB is smaller than the sum of the areas of the circular sector OAC and the triangle ACB. This can be written as $\frac{t}{2}+\operatorname{tg}t\sin^2\frac{t}{2}>\frac{\operatorname{tg}t}{2}$, so by using $\operatorname{tg}t>2\sin\frac{t}{2}$ $(0<t<\frac{\pi}{2})$, we obtain $t>\operatorname{tg}t-\frac{\operatorname{tg}^3 t}{2}$ and this implies $x-\frac{x^3}{2}<\operatorname{arctg}x$.

From (2) we obtain

$$\operatorname{arctg}\frac{n}{n^2+k^2+k}<\frac{n}{n^2+k^2+k}<\frac{n}{n^2+k^2}\quad\text{and}$$
$$\frac{n}{n^2+k^2+k}-\frac{1}{2}\left(\frac{n}{n^2+k^2+k}\right)^3<\operatorname{arctg}\frac{n}{n^2+k^2+k}.$$

Using a short straightforward calculation we can verify that

$$\frac{n}{n^2+k^2+2k+1}<\frac{n}{n^2+k^2+k}-\frac{1}{2}\left(\frac{n}{n^2+k^2+k}\right)^3.$$

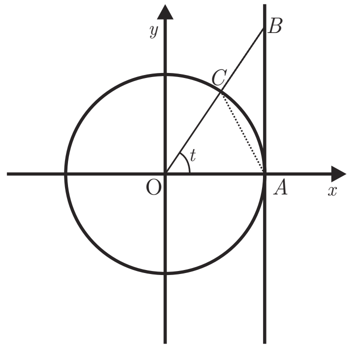

Fig. 2. Inequalities for tg t

Indeed, this inequality is equivalent (after reducing the fractions) with

$$n^2(n^2+k^2+2k+1) < 2(k+1)(n^2+k^2+k)^2$$

and this holds because $n^2(n^2+k^2+2k+1) < (x-y)(x+y) < x^2 < 2yx^2$, where $x = n^2+k^2+k$ and $y = k+1$. So

$$\operatorname{arctg}\frac{n}{n^2+(k+1)^2} < \operatorname{arctg}\frac{n}{n^2+k^2+k} < \frac{n}{n^2+k^2}.$$

From these inequalities we obtain the following estimations

$$x_n = \sum_{k=1}^{n}\frac{n}{n^2+k^2} > \sum_{k=1}^{n}\operatorname{arctg}\frac{n}{n^2+k^2+k} = \operatorname{arctg}\frac{n+1}{n} - \operatorname{arctg}\frac{1}{n},$$

$$x_n = \sum_{k=1}^{n}\frac{n}{n^2+k^2} < \sum_{k=0}^{n-1}\operatorname{arctg}\frac{n}{n^2+k^2+k} = \operatorname{arctg}\frac{n}{n}.$$

On the other hand $\lim_{n\to\infty}\operatorname{arctg}\frac{n+1}{n} - \operatorname{arctg}\frac{1}{n} = \operatorname{arctg} 1 = \frac{\pi}{4}$, so $\lim_{n\to\infty} x_n = \frac{\pi}{4}$. ∎

In what follows we give some sufficient conditions for the monotonicity of Riemann type sums constructed as above.

Theorem 1. *If the function $f : [0,1] \to \mathbb{R}$ is concave and decreasing, then the sequence*

$$a_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

is increasing and the sequence

$$b_n = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

is decreasing.

Proof. We prove that

$$\left(\frac{k+1}{n+1} - \frac{k}{n}\right)\left(f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k+1}{n}\right)\right) \geq \left(\frac{k}{n} - \frac{k}{n+1}\right)\left(f\left(\frac{k}{n}\right) - f\left(\frac{k+1}{n+1}\right)\right). \tag{3}$$

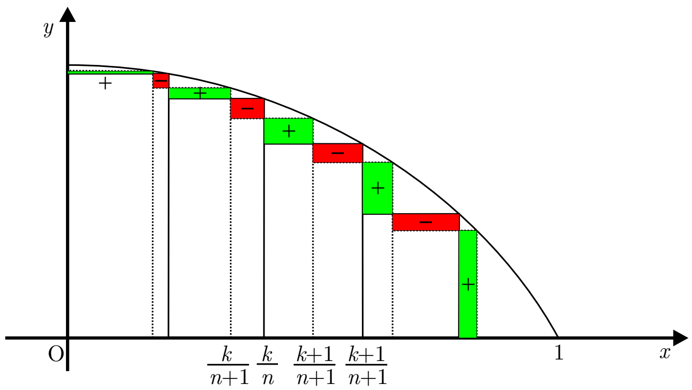

Fig. 3. Subdivision for n and $n+1$ for a concave function

This is equivalent to the following inequalities

$$\left(\frac{k+1}{n+1}-\frac{k}{n+1}\right)f\left(\frac{k+1}{n+1}\right)\geq\left(\frac{k+1}{n+1}-\frac{k}{n}\right)f\left(\frac{k+1}{n}\right)+\left(\frac{k}{n}-\frac{k}{n+1}\right)f\left(\frac{k}{n}\right)$$

$$f\left(\frac{k+1}{n+1}\right)\geq\frac{n-k}{n}f\left(\frac{k+1}{n}\right)+\frac{k}{n}f\left(\frac{k}{n}\right).$$

On the other hand f is concave, so we have

$$f\left(\frac{n-k}{n}\cdot\frac{k+1}{n+1}+\frac{k}{n}\cdot\frac{k}{n}\right)\geq\frac{n-k}{n}f\left(\frac{k+1}{n}\right)+\frac{k}{n}f\left(\frac{k}{n}\right),$$

hence

$$f\left(\frac{nk+n-k}{n^2}\right)\geq\frac{n-k}{n}f\left(\frac{k+1}{n}\right)+\frac{k}{n}f\left(\frac{k}{n}\right). \tag{4}$$

Due to the monotonicity of the function f and the inequality

$$\frac{nk+n-k}{n^2}\geq\frac{k+1}{n+1},$$

we have

$$f\left(\frac{k+1}{n+1}\right)\geq f\left(\frac{nk+n-k}{n^2}\right). \tag{5}$$

From (4) and (5) we obtain (3) and this implies $a_{n+1}\geq a_n$.

The second part of the proof can be obtained by using a similar argument or we can simply change the orientation of both axis and placing the origin to $(1,f(0))$ in order to reduce this part of the proof to the first part of the following theorem. ■

Theorem 2. *If the function $f:[0,1]\to\mathbb{R}$ is convex and decreasing, then the sequence*

$$a_n=\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)$$

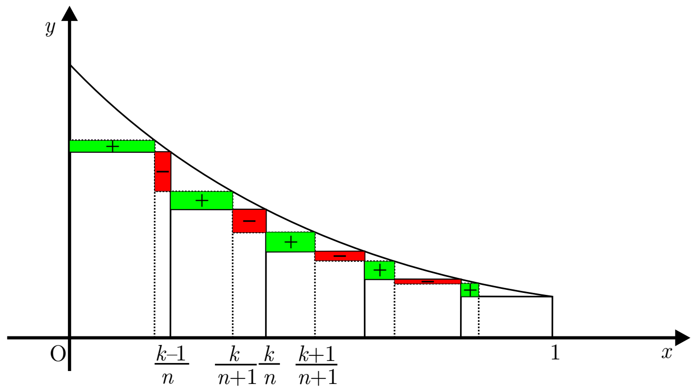

Fig. 4. Subdivisions for n and $n+1$ for a convex function

is increasing and the sequence

$$b_n = \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

is decreasing.

Proof. We prove that

$$\left(\frac{k}{n+1} - \frac{k-1}{n}\right)\left(f\left(\frac{k}{n+1}\right) - f\left(\frac{k}{n}\right)\right) \geq \left(\frac{k}{n} - \frac{k}{n+1}\right)\left(f\left(\frac{k}{n}\right) - f\left(\frac{k+1}{n+1}\right)\right). \tag{6}$$

This is equivalent to the following inequalities

$$\left(\frac{k}{n} - \frac{k-1}{n}\right) f\left(\frac{k}{n}\right) \leq \left(\frac{k}{n+1} - \frac{k-1}{n}\right) f\left(\frac{k}{n+1}\right) + \left(\frac{k}{n} - \frac{k}{n+1}\right) f\left(\frac{k+1}{n+1}\right)$$

$$f\left(\frac{k}{n}\right) \leq \frac{n+1-k}{n+1} f\left(\frac{k}{n+1}\right) + \frac{k}{n+1} f\left(\frac{k+1}{n+1}\right).$$

From the convexity and monotonicity of f we have

$$f\left(\frac{n+1-k}{n+1} \cdot \frac{k}{n+1} + \frac{k}{n+1} \cdot \frac{k+1}{n+1}\right) \leq \frac{n+1-k}{n+1} f\left(\frac{k}{n+1}\right) + \frac{k}{n+1} f\left(\frac{k+1}{n+1}\right) \tag{7}$$

and

$$f\left(\frac{(n+2)k}{(n+1)^2}\right) \geq f\left(\frac{k}{n}\right), \tag{8}$$

because

$$\frac{n+1-k}{n+1} \cdot \frac{k}{n+1} + \frac{k}{n+1} \cdot \frac{k+1}{n+1} = \frac{(n+2)k}{(n+1)^2} \leq \frac{k}{n}.$$

Inequalities (7) and (8) imply (6) and from this we obtain $a_{n+1} \geq a_n$.

The second part of the proof is equivalent to the first part of the previous theorem or can be obtained using a similar argument. ∎

3. Concluding remarks

1. Based on students' former knowledge the emphasized problems are natural and can come up during teaching activities. The difficulties which we have to face in solving such kind of problems motivates (and had motivated throughout the history of mathematics) the introduction of a slightly different point of view in the construction of the Riemann integral.

2. In the existing curricula Problem 3 (and many other similar problems) is solved using integrals, so students do not have the occasion to understand that the use of the integral is a very powerful and effective way for solving these kind of problems and in fact spares us from individual effort in treating these problems.

3. The presented problems and solutions offer us a good example for the first statement of the motto and in the same time an insight into understanding why is convenient to choose the second statement as teaching attitude in many cases. In order to be successful in teaching mathematics and science we need to fight against this attitudes as many times as it is possible for us.

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