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AN APPROACH TO INCORPORATE DYNAMIC GEOMETRY SYSTEMS IN SECONDARY SCHOOL —MODEL WITH MODULE

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Abstract. The paper presents the key elements of a model for in-service teachers training. An educational module about parabola as geometry object is the illustrative part of the model. The inclusion of dynamic geometry system applets allows some properties of the parabola to be established by examining dynamic constructions. Starting with the focusdirectrix definition, the reflective property of the parabola has been proven and the equation of the parabola is worked out. A comprehensive comment on doubling-the-cube problem is given. Some directions for further development of the topic are pointed out.

ZDM Subject Classification: D44, U54; *AMS Subject Classification*: 97D40, 97E50.

Key words and phrases: Dynamic geometry; parabola.

0. Introduction

Our recent experience in training mathematics teachers shows that some of the teachers who are excellent experts in traditional mathematics teaching feel discomfort and lack of self-confidence when the matter comes to incorporate computer-based parts in the lesson. The efforts to train the teachers in computer skills turn things from bad to worst: their uncertainty grows. However, to avoid computer technologies in mathematics teaching because of staff problems do not meet the spirit of the times. So we designed a model for teacher training that partially improves the situation. The paper presents the framework of the model and a module for illustrating the main points of the model.

1. The framework of the context model

The core of our context model is teacher training process organized by educational modules of flexible type [1]. The module presents a topic in a mixed mode: traditional paper-and-pencil technique and dynamic geometry applets. According to a particular classroom context, the teacher decides how to organize the teaching. The structure of the module allows applying elements from the teaching-in-context model [2]. We offer five activity types for incorporating any applet: *illustration, demonstration, construction, deduction*, and *application* (definitions are given in the Appendix). It is important to note that when the teacher enters the classroom (s)he has all the applets prepared in advance.

However, the case is quite different during the teachers training: teachers are often urged to make applets by themselves and to get an idea of the difficulty of the task. The ready-made applets included in the module are as simple as possible and their mathematical background is clearly seen. Such design allows teacher to feel confidence in potential answering questions about details in construction the teacher use during the lesson.

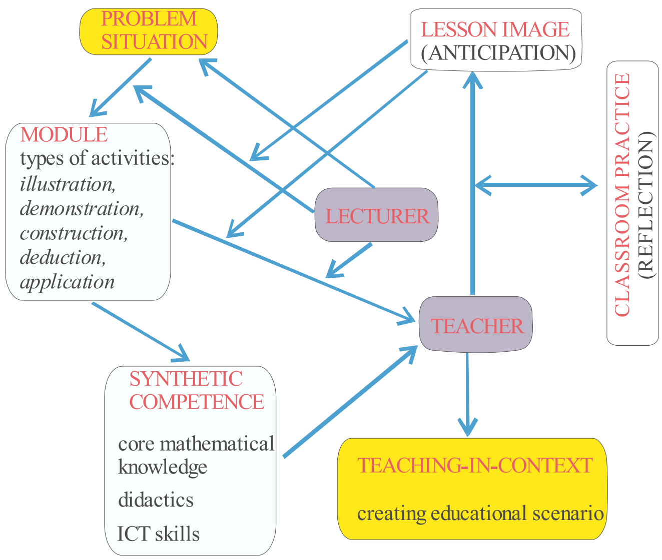

Fig. 1

The sketch in Fig.1 gives idea of how the training of the teachers is expected to go. A very important detail in the context model is that the final decision of how to use any of the applets is taken by the teacher. Another important feature is that the interaction between the lecturer and the learner excludes the instructional type of training. Further we will clarify by examples the other details and connections of the model.

2. Choosing the topic

The topic under consideration in the module should be carefully selected. On the one hand, it should be of significant importance, but also the matter should be available for both teachers and students. On the other hand the topic should motivate using of DGS, i.e., the applets should provide considerable progress in studying compared with the traditional ways of teaching. In our opinion the topic parabola meets the above criteria.

Parabola is presented in Bulgarian secondary school mathematics mainly as the graph of the quadratic function, which does not allow establishing any properties of the curve. The neglect of the geometrical nature of the parabola in secondary school comes possibly from the limited traditional resources. Below we give an idea of how to correct such a misleading view point. A crucial role play the implementation of DGS applets that are designed with GeoGebra but this could be done with any other DGS. Modern dynamic software allows to overcome the frame of traditional teaching-learning process, i.e. time-limitation and visualization obstacles. However, we prefer miscellaneous techniques that give students some space for DGS exploration but also including tempered conservative paper-and-pencil activities mainly in deduction.

Parable (in literature) is a brief, succinct story, which illustrates a moral or religious lesson. We adopt a bit of this style that usually drags instructional teaching to contrast the inquiry based approach of the content. The idea is to remind that exploration and heuristics in mathematics education could be effective only on deductive fundament.

3. The module Introducing Parabola

3.1 Prologue

The name parabola is due to Apollonius [3]. The focus-directrix property of the parabola (which we adopt as definition) is due to Pappus [4].

Definition 1. Given a point F and a line d not passing trough F. The locus Π of the points P that are equidistant to F and d is a curve called *parabola*. The line d is called *directrix* and the point F is called *focus* of the parabola Π. The perpendicular line a to d through F is called *axis* of the parabola and the point of Π that lies on a is called vertex of the parabola.

Challenge 1. Given an arbitrary segment m, construct a point M at distance m to both d and F, by ruler and compass. (Solutions of the Challenges are given in the Appendix.)

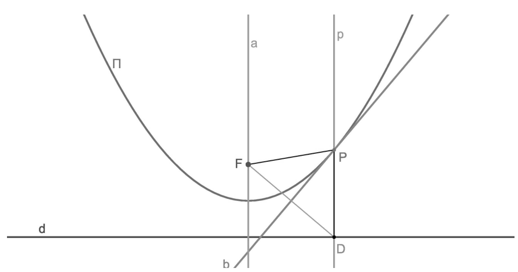

Fig. 2

Dynamic construction. 1) Take an arbitrary point D on d and erect the perpendicular p to d through D.

2) Draw the perpendicular bisector b of FD.

3) Label by P the intersection point of p and b. (Fig. 2)

4) Moving D along d we generate points from the parabola Π (the continuous trace of P is the whole parabola.

Proof of 4). Since P lies on b (we write Pzb), it is equidistant to F and D, i.e., $PF = PD$. Since Pzp, the distance from P to d equals PD. Thus, P is equidistant to F and d.

Further we will use by default the notations from the above construction.

CHALLENGE 2. Let l be a line parallel to d that intersects Π. Let L be the foot of the perpendicular through P to l. Explore $FP + PL$ when P moves along Π, staying between d and l. Prove that $FP + PL$ does not depend on P in this case (Fig. 3).

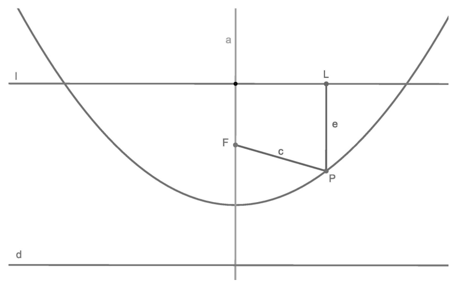

Fig. 3

3.2 Reflection property of the parabola

PROPOSITION 1. *The point P is the only common point of Π and b.*

Proof. Suppose the contrary. Let P' be another common point of Π and b. Since $P'zb$, then $P'F = P'D$. Consider the perpendicular p' to d through P' whose foot on d is D' and let b' be the segment bisector of FD'. Since $P'z\Pi$, it is equidistant to F and d, which gives $P'F = P'D'$ and hence $P'zb'$. Therefore, $P'D = P'F = P'D'$, i.e. P' is equidistant to D and D'. This is a contradiction because the triangle $P'D'D$ is right-angled at D'.

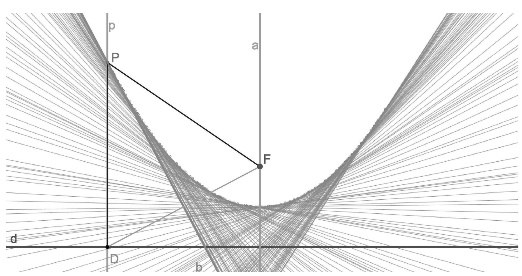

Fig. 4

Proposition 1 leads to another representation of our parabola Π as the *envelope* of the family of lines b that are the perpendicular bisectors of the segments FD when D moves along d. This envelope could be seen in our dynamic construction by turning on the trace option for b on our dynamic construction (Fig. 4).

Definition 2. A line that has only one common point with a parabola is called *tangent* to this parabola.

Proposition 2. *The perpendicular bisector b is a tangent line to* Π.

There is only one tangent line through any point of the parabola (called *point of tangency*). Our construction guaranties the existence of a tangent through any point of the parabola but here we are not able to prove its uniqueness. Further, when we talk about tangent line to the parabola, we will mean the one constructed above (i.e. lines b).

Proposition 3. *The lines PF and PD meet the tangent b at equal angles.*

Proof. Since $\triangle FDP$ is isosceles one with $PF = PD$, then the perpendicular bisector b of FD is the angle bisector of angle FPD.

Proposition 4. *A light ray coming from F is reflected by the tangent to the parabola at the tangency point P and turns into a ray that is parallel to the axis of* Π. (Keep in mind the *Law of Reflection*: the angle of reflection equals the angle of incidence.)

Proof. Let $PQ^{\rightarrow}$ be the ray of reflection of the ray $FP^{\rightarrow}$ (Fig. 5). Since the angle between PF and b equals the angle between PQ and b, then (as Proposition 3 says) it equals the angle between PD and d. Hence, the last two angles are vertically-opposite, which gives that $PQ^{\rightarrow}$ and $PD^{\rightarrow}$ are opposite rays. Thus $PQ \perp d$ and so does the axis a. Therefore $a \parallel PQ^{\rightarrow}$.

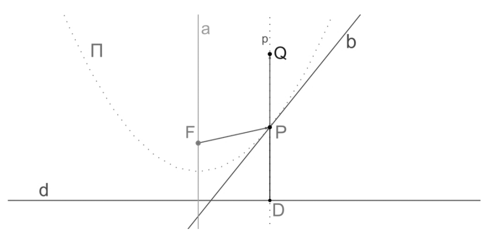

Fig. 5

A reflecting curve reflects an incident ray at a point in the same way as the tangent line at the same point would reflect such a ray. In fact *the parabola reflects the incident rays in the manner the envelope of its tangents does.* This means that Proposition 4 could be reformulated as: *A light ray leaving the focus after being reflected by the parabola turns into a ray which is parallel to the axis of the parabola* (Fig. 6).

The above statement is known as the *reflective property of parabola*. In 3D case the *reflecting surface* takes the role of the curve and the *tangent plane* takes the role of the tangent line. Having these facts one can interpret the name parabola from the reflective-property perspective.

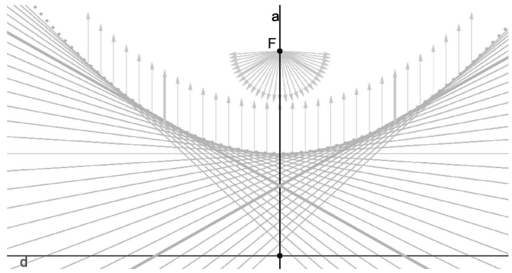

Fig. 6

3.3 Equation of the parabola

Coordinatization. Let the vertex of Π be the origin of the (Cartesian) coordinate system whose y-axis lies on a. Let the coordinates of P be $(x; y)$. If the distance from F to d is $2q$, then $F(0; q)$. The distance from P to d is $y + q$ and by the definition of Π it follows that $PF = y + q$ (Fig. 7).

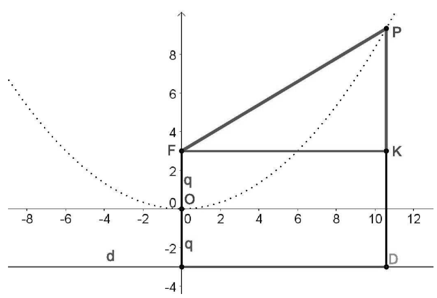

Fig. 7

Working out the equation. Let $K(x; q)$ be as shown in Fig. 7. By the Pythagorean Theorem for $\triangle FKP$ we obtain consecutively

$$FK^2 + KP^2 = PF^2,$$
$$x^2 + (y - q)^2 = (y + q)^2,$$
$$x^2 + y^2 - 2qy + q^2 = y^2 + 2qy + q^2,$$
$$y = \frac{1}{4q}\,x^2.$$

The last equation is the *canonical equation of the parabola*.

Challenge 3. How does the canonical equation of the parabola change when the origin O is taken: a) in the projection of F onto d; b) in F?

Challenge 4. Prove that there is just one tangent to the parabola at any of its points.

3.4 Doubling the cube

The problem owes its name to a story concerning the citizens of Delos, who consulted the oracle at Delphi in order to learn how to defeat a plague sent by Apollo. The oracle responded that they must double the volume of the altar to Apollo, which was in the shape of a cube [3]. Denote by m the edge of the existing altar, the edge n of the new altar should satisfy the equation $n^3 = 2m^2$. So we come to the following

Problem. Given an arbitrary segment m, construct the segment $n = m\sqrt[3]{2}$.

This problem has no solution by ruler and compass, starting from blank plane [5]. However, having a parabola drawn, one cane manage to solve it.

Non-dynamic construction. Given an arbitrary segment m, consider the Cartesian coordinates with m as unit segment. Given is the parabola Π with canonical equation $y = x^2$, i.e., F and d are at distance $1/4$ from the origin O and $d \parallel Ox$.

1) Draw the circle k centered at $C(1; 0.5)$ and passing through O.

2) Mark the intersection point P of Π and k and draw the line p through P, perpendicular to the abscissa.

3) Mark the projection N of P on the abscissa and draw the segment $n = ON$ – the desired segment (Fig. 8).

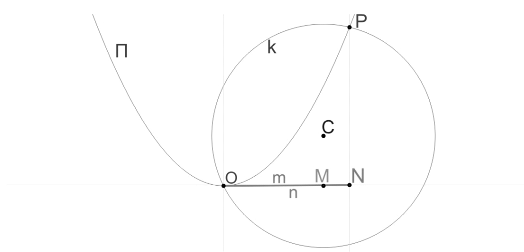

Fig. 8

Proof. The radius CO of k equals $\sqrt{5}/2$. For any point $X(x; y)$ of k the equation $(x-1)^2 + (y-0.5)^2 = 1.25$ holds. The real solutions of the simultaneous equations

$$y = x^2$$
$$(x-1)^2 + (y-0.5)^2 = 1.25$$

are $(0; 0)$ and $(\sqrt[3]{2}; \sqrt[3]{4})$. Thus $N(\sqrt[3]{2}; 0)$ and $n = ON = \sqrt[3]{2}$.

Our non-dynamic construction leads to a construction of the segment $n = m\sqrt[3]{2}$ by ruler and compass. This construction cannot omit the getting of Π as connected line. The dynamic construction gives as many points of Π as we wish, but not all the points of Π, i.e., represents a discrete part of Π. By solving the above challenge we do construct the segment $n = m\sqrt[3]{2}$ by ruler and compass, but we do need an arc from the parabola Π as connected curve in advance.

4. How the context model works

Further we are going to discuss some details in implementation of the context model. We also give the first feedback results and some possible directions to upgrade the module.

4.1 Didactical parameters

The module *Parabola* is oriented to 9th–12th grade advanced students in mathematics and sciences. A priori time consumption is about 90 minutes, which corresponds the single extracurricular lesson. Students are expected to have idea about loci, Pythagorean Theorem, 2D coordinates, reflection, quadratic systems, algebraic inequalities, ruler-and-compass axioms, as well as initial skills in DGS (e.g., GEONExT or GeoGebra) and their teachers should be familiar with the listed topics.

4.2 Some statistics

As it was pointed, according to the classroom context the DGS inclusion could be of the following activity types: *illustration, demonstration, construction, deduction*, and *application*. An inquiry held in the frame of a training course in June 2010 in Bulgaria, gives the following statistics for how the teachers plan to include the Challenge 2 in their practice:

Total entries 21;

as demonstration – 4;

as construction – 3;

as demonstration and deduction – 6;

as construction and deduction – 1;

as illustration or demonstration or construction – 1;

as illustration or demonstration and deduction – 2;

no idea – 1.

The variety of answers comes from the flexibility of the teachers' anticipation (beliefs) but also because DGS applets provide enough space for such flexibility. Our impressions were that teachers overcome their initial fright of DGS and they are ready to put into teaching practice the module Parabola. As far as we know (June, 2011) there were two successful attempts to organize teaching based on the Module Parabola: at National Mathematical High School with 12th grade students in 2010 and at Sofia Mathematical High School with 10th grade students in 2011.

4.3 Examining a DGS construction

The way of using Challenge 2 during the classes at Sofia Mathematical High School was of type *demonstration* and *illustration* with *deduction*. Students from one of the groups were given the ready applet and they were encouraged to observe the behavior of $FP + PL$ with respect to P after changing the location of l, F and d consecutively. The other group was given to prove the assertion without any exploration of the construction. After proving the $FP + PL$ stays invariant with respect to the location of P, some 'evidences' in dynamic style were shown. Students in both groups reacted similarly.

4.4 Teachers' creativity

The context model we apply in teachers training encourages teachers to interpret the existing applets from unexpected (for the author) perspective or/and to develop their own DGS applets. The applet on Fig. 9 is designed by us after an idea of G. Genchev, a teacher from Dimitrovgrad, who took part in a training course in April 2010.

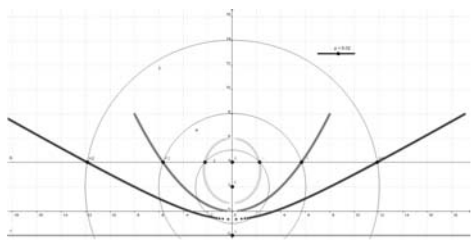

Fig. 9

Teachers in physics from Pravez interpreted the same applet as wavefronts during a training course in May 2010. (By the way, they were rather satisfied of having proof of the reflective property of the parabola which they present to students only as experimental fact.)

4.5 Control

The teacher can control the learning process either giving advices, or giving marks (preferably qualitative), or even leaving students to evaluate themselves—it also depends on the particular classroom context. Such supervising role does not require the teacher's advanced skills in dynamic geometry system (DGS). Let us note that the DGS also has a kind of control function—usually when a dynamic construction is not properly designed (by the students) it does not work properly.

4.6 DGS inconvenience

Proving from the contrary often needs drawing far from accuracy (a kind of imperfection) which is not possible to manage with DGS. E.g., a sketch that illustrates the proof of Proposition 1 must be made by hand.

4.7 Going further

Introduction of the concept of envelope is extended in another module [6]. Some more geometry problems about parabola are available on the page `http://www.cut-the-knot.org/ctk/Parabola.shtml`. Fig. 9 is designed to illustrate the focus-directrix-eccentricity definition of parabola, ellipse and hyperbola. We cannot see motivation to introduce ellipse and hyperbola in Bulgarian secondary school mathematics curriculum, so the topic is left to the teachers who can find reasons to do this.

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Appendix

A1. Definitions of activity types in the module

Illustration could be a picture, figure, animation, prepared in advance; the DGS is used by the teacher as drawing tool; learners could establish connections between objects by observation.

Demonstration is a dynamic applet prepared in advance; learners explore the relations between objects, locate invariants etc.; following some clues they can "discover" some facts about the objects on the picture.

Construction is a picture or a dynamic applet, which is expected to be constructed by the learners; the DGS applet should visualize relations between some mathematical objects and a proper hierarchy of these objects is necessary to be stated.

Deduction is a detached part of the module in which learners are expected to summarize their observations and to argue their conclusions in traditional deductive style (ICT=free part of the module).

Application is the moment of true for IBSME—learners are encouraged to check their competence in problems that go beyond the routine practice; they can state and check conjectures, prove results, propose argued ideas for constructions related to the theme; this is the activity that requires learners' full scale synthetic competence.

A2. Solutions of the Challenges in the module

Solution of Challenge 1. 1) Draw the lines $l_{1,2}$ parallel to d at distance m to d.

2) Draw the circle k with radius m centered at F.

3) Take M as an intersection point of k and $l_{1,2}$.

Solution of Challenge 2. Denote by D the foot of the perpendicular through P to d. From $Pz\Pi$ we have $PF = PD$. Now $PF + PL = PD + PL = DL$. But DL is the distance between d and l thus it does not depend on the location of P.

Solution of Challenge 3. a) If $FO = 2q$ then $y - q = \frac{x^2}{4q}$. Thus the equation of the parabola is $y = \frac{x^2}{4q} + q$.

b) If the distance between d and O is $2q$ then $y + q = \frac{x^2}{4q}$. Thus the equation of the parabola is $y = \frac{x^2}{4q} - q$.

Solution of Challenge 4. Suppose the line t: $y = mx + n$ is tangent to the parabola Π: $y = \frac{x^2}{4q}$ at the point $P(u, v) z \Pi$. By the definition of tangent to parabola there is just one common point of Π and t. Therefore (u, v) is the only solution of the simultaneous equations $y = mx + n$, $y = \frac{x^2}{4q}$. From here we conclude that u is the only root of the quadratic equation $x^2 - 4qmx - 4qn = 0$. Thus $u = 2qm$, i.e., $m = \frac{u}{2q}$. But there is a unique line through $P(u, v)$ with slope $\frac{u}{2q}$ which proves the assertion of the Challenge.