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EULER FORMULA AND MAPS ON SURFACES

Siniša T. Vrećica

Dedicated to Milo Marjanović, my topology teacher and friend, on the occasion of his 80th birthday

Abstract. This paper is addressed primarily to those high school students with an intensive interest in mathematics, who are often in search for some extra reading materials not being on school curriculum, as well as to their teachers. We have chosen to offer here a material elaborating how the Euler formula could be used to establish non-planarity of some graphs, and results about the colorings of maps on surfaces.

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1. Introduction

The Euler characteristic is a topological notion appearing in many different topics throughout mathematics. The Euler formula shows that the expression of the Euler characteristic in terms of numbers of vertices, edges and regions (faces of different dimension in higher-dimensional cases) is a topological invariant. This formula is a powerful tool used in establishing many important mathematical results, from the classification of regular polyhedra to the non-planarity criterium for graphs.

This formula is also used, among many other ideas, in [3] by M. Marjanović and his co-authors in their approach to the problem of recognizing the shape of figures (of letters, in particular).

We intend to describe how this result is used to determine the minimal number of colors needed to color any map in the plane, or more generally, on any given surface.

2. Graphs and maps

Let us start by introducing some necessary notions.

By an *(undirected, finite) graph* we will mean an ordered pair $G = (V, E)$, where the elements of the finite set V are called *vertices*, and elements of the finite set E are called *edges*, and each edge corresponds to a pair of vertices. (We think of the edges as intervals joining the corresponding two vertices.) Loops and multiple edges among two vertices are allowed. For the reader introduced to the notion of simplicial complex, a graph without loops and multiple edges is a 1-dimensional simplicial complex.

Let us also mention some special classes of graphs. By K_n we will denote the complete graph with n vertices, where each pair of different vertices is joined by an edge. By $K_{m,n}$ we will denote a complete bipartite graph on two groups of m and n vertices respectively, where each vertex from the first group is joined to each vertex of the second group, and no two vertices from the same group are joined.

A *map* on some surface S (including the plane) is an injective continuous mapping $\varphi\colon G \to S$ from some graph G to S. With no ambiguity, the images of the vertices are also called vertices, and the images of the edges are also called edges. So, edges of the map are some "curves" on surface S with no self-intersection, each two of which could intersect only at possible common vertex. We could think of a map as of some "curved" image of a graph embedded in the surface.

If there is a map of the graph G to surface S, we say that graph G could be embedded in S. If a graph could be embedded in the plane, than it could be embedded in the sphere as well, since the sphere without one point is homeomorphic to the plane. The vice-versa is true as well. Namely, if there is an embedding of a graph in the sphere, there is a point on the sphere not belonging to the image of the embedding. (Otherwise this embedding would be a homeomorphism, which of course could not exist.) But, then a graph could be embedded in the sphere without a point, which is homeomorphic to the plane. We call such graphs *planar*.

The connected components of the complement of the map are called its *regions*, and each map determines a finite number of regions.

If we consider a map in the plane, then some of its regions are unbounded and the other regions are bounded. On every closed surface (compact, without boundary), all regions have compact closures.

We will be especially interested in the maps on closed surfaces whose regions are homeomorphic to an open disc. Note that this forces the graph G to be connected. In the case of the plane (the unique remaining case that we will consider), there is one unbounded region which could not be homeomorphic to an open disc, and the graph G need not to be connected. In this case we impose the additional condition to the graph G to be connected. (The readers familiar with some notions of topology, could easily see that this is equivalent to the requirement that the unbounded region is homotopy equivalent to the circle. We choose to require the graph G to be connected, in order to avoid the usage of the notion, which might be not familiar to some readers.) In this text we reduce our attention to such maps.

If we consider the image of a bipartite graph, then each region is bounded by at least four vertices and edges.

Each map $\varphi\colon G \to S$ determines a graph G^* (in some sense) dual to the graph G, where the vertices of G^* correspond to the regions of the map, and two vertices of G^* are joined if and only if corresponding regions share a common edge on their boundaries. Note that this dual graph G^* comes with a natural embedding in the surface S. Note also that this notion depends on a map of the graph G, and not only on the graph G itself. It is obvious that a dual graph contains no loops or multiple edges by definition.

Under some assumptions, there would be a construction in the opposite direction. Starting from the graph G^* embedded in S, there would be such graph G and a map $\varphi\colon G \to S$ so that the dual graph is exactly the original graph G^*. However, we will not use it here.

3. Euler characteristic

Given a graph $G = (V, E)$ with v vertices and e edges, its *Euler characteristic* is defined to be $\chi(G) = v - e$.

For a graph G embedded in the plane, the following idea was used in [3] to simply determine its Euler characteristic. In order to simplify the presentation, we present the idea in the case of the graph with no loops, although it works in general.

Let us choose a line l in that plane, and project the drawing of G to the line l. We could choose a generic line l in the plane, so that no two vertices of G project to the same point in l, and so that every point x on l is the projection of finitely many, $n(x)$, points from G. This function $n(x)$ is constant in some intervals and let us denote by $a_0, a_1, \ldots, a_k$ the successive endpoints of these intervals in one direction. These are the points of the line l in which function $n(x)$ changes its value. We want that every point which is projected to some of these points is a vertex of G. If some point which is not a vertex projects to some a_i, we could introduce it as a vertex and split the edge containing it in two edges. In this process we increase the number of vertices and edges by 1 and so, preserve the Euler characteristic.

If some vertex projects to a point in some open interval (a_i, a_{i+1}), it is an endpoint of two edges, and such a vertex could be removed from the set of vertices by joining these two edges in one edge (in a kind of reverse process). Notice that the Euler characteristic is again preserved, since we decrease the number of vertices and edges by 1.

By applying these two processes we obtain an embedded graph whose vertices (and only vertices) are all projected to the points $a_0, a_1, \ldots, a_k$. So, the number of vertices of this graph is $\sum_{i=0}^{k} n(a_i)$. Also, the vertices of every edge are projected to two consecutive points in the order $a_0, a_1, \ldots, a_k$.

Let us now choose some points $b_1 \in (a_0, a_1), \ldots, b_k \in (a_{k-1}, a_k)$ arbitrarily in these intervals. Each point projected to the point b_i belongs to an edge connecting two vertices which are projected to the points a_{i-1} and a_i. Different points projected to the same point b_i correspond to different edges of G, and so $n(b_i)$ counts the number of edges whose vertices project to the points a_{i-1} and a_i. So, the number of edges of this graph is $\sum_{i=1}^{k} n(b_i)$, and we summarize all this in the following theorem.

Theorem 3.1. $\chi(G) = \sum_{i=0}^{k} n(a_i) - \sum_{i=1}^{k} n(b_i)$.

Let us now consider a map $\varphi\colon G \to S$ on the closed surface S, with v vertices, e edges, which determines r regions, all of which are homeomorphic to a disc.

The Euler characteristic of this map is defined to be $\chi(\varphi) = v - e + r$.

In the case of the plane we denote by r the number of bounded regions (so we do not count one unbounded region), and also define the Euler characteristic in the same way as $\chi(\varphi) = v - e + r$. In this case, we require that all regions except for the unbounded one are homeomorphic to the disc, and that graph G is connected.

The Euler-Poincaré formula expresses this quantity in terms of Betti numbers of the surface S (their alternating sum). As a consequence, it turns out that the Euler characteristic is a topological invariant of the surface, and it does not depend on a map but only on the surface. We denote it by $\chi(S)$.

We provide an elementary proof of this fact. Let $\varphi_1\colon G_1 \to S$ and $\varphi_2\colon G_2 \to S$ be two maps on the surface S whose regions are homeomorphic to a disc (except for the unbounded one in the case of the plane). We want to show $\chi(\varphi_1) = \chi(\varphi_2)$. Without lack of generality we could assume that no vertex of one map belongs to any edge of the other map, and that edges of different maps intersect transversally. (This requirement could be easily fulfilled by small perturbation of one map.) Similarly, we could require, without lack of generality, that at least one edge of φ_1 intersects some edge of φ_2, i.e., that neither of these maps is contained in one region determined by the other map.

Let us denote by φ "the union" of these two maps. More precisely, let the vertices of φ be the vertices of φ_1, the vertices of φ_2, and the intersection points of an edge of φ_1 and an edge of φ_2. These intersection points (new vertices of φ) will subdivide the edges of φ_1 and φ_2 in the edges of φ. It is easy to see that we could obtain the map φ from the map φ_1 (or φ_2) in the finitely many steps of the following three types:

(i) Add a vertex in the interior of some edge and divide this edge in two edges. In this step the number of vertices and edges increase both by 1, and so the Euler characteristic does not change;

(ii) Add a vertex in the interior of some region and connect it to one vertex of that region. In this step, also, the number of vertices and the number of edges increase by 1. Again, the Euler characteristic does not change;

(iii) Connect two vertices of some region by an edge. In such step the number of edges and the number of regions increase by 1. So, the Euler characteristic does not change in such step, either.

Consequently, $\chi(\varphi) = \chi(\varphi_1)$. In the same way we obtain $\chi(\varphi) = \chi(\varphi_2)$, which proves our claim.

An alternative elementary proof of this fact in the case of the sphere could be found in [4].

As a consequence, we could determine the Euler characteristic of some surface S by counting the number of vertices, edges and regions of some conveniently chosen map on S. It is a trivial exercise for a reader now to see that $\chi(\mathbb{R}^2) = 1$ and $\chi(S^2) = 2$.

It is well known that every oriented closed surface is homeomorphic to some surface M_g (the sphere with g handles, or equivalently, connected sum of g tori M_1), $g = 0, 1, 2, \ldots$. Of course, M_0 is a sphere, M_1 is a torus (surface of a doughnut), and M_2 could be visualized like this:

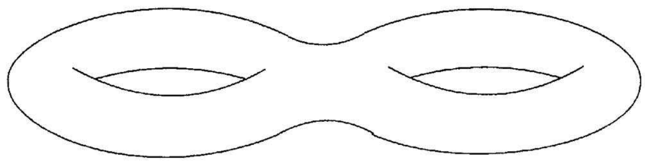

Fig. 1

The surface M_g could be obtained from the wedge of $2g$ circles $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ by gluing $4g$ pieces of the boundary of a disc to these circles along the "word" $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}$. (Note that in the case of the sphere $M_0 = S^2$ there are no circles, and the boundary of a disc is glued to a point.) So, the Euler characteristic of M_g could be determined as $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$. The same answer could be obtained by computing the Betti numbers of M_g.

Every non-oriented closed surface is homeomorphic to some of surfaces N_h (connected sum of h projective planes), $h = 1, 2, \ldots$. These surfaces are difficult to visualize since they cannot be embedded in the three-dimensional space. However, the reader is encouraged to visit the web-page `http://vimeo.com/22409616` (Vimeo) or the web-page `http://www.youtube.com/watch?v=9gRx66xKXek` (You Tube), at which he/she could view the animation by Dušan Živaljević, in order to get some geometric insight in this sequence of surfaces, the projective plane $N_1 = \mathbb{R}\mathrm{P}^2$. This animation contains the description of the embedding of the projective plane in $\mathbb{R}^3$ with "transversal" self-intersection.

The surface N_h could be obtained from the wedge of h circles $\gamma_1, \ldots, \gamma_h$ by gluing $2h$ pieces of the boundary of a disc to these circles along the "word" $\gamma_1^2 \cdots \gamma_h^2$. So, the Euler characteristic of N_h could be determined as $\chi(N_h) = 1-h+1 = 2-h$. Again, the same answer could be obtained by computing the Betti numbers of N_h.

Remark. We could also consider maps not satisfying the properties that the regions are homeomorphic to open discs and that the graph is connected. In this case the quantity $v - e + r$ depends on a map. However, it could be verified that, by the above steps of type (i), (ii) and (iii), the original map could be turned into a map whose regions are homeomorphic to open discs (except for one in the case of plane), the resultant graph is connected, and such that the quantity $v - e + r$ does not increase. The only difference in this case is that in the step of type (iii) the number of regions could remain the same, and so the quantity $v - e + r$ decreases.

Consequently, for any map on S we have $v - e + r \geq \chi(S)$.

4. Planarity of graphs

The problem of characterizing the planar and non-planar graphs was one of very important and well-known mathematical problems until it was solved almost simultaneously by Kuratowski and Pontryagin at the end of 1920's. Their result says that a graph is non-planar if and only if it contains one of the graphs K_5 and $K_{3,3}$ as a minor (more geometrically, if its drawing contains a drawing of one of these graphs). This result is beyond the scope of this text, and here we prove only the easier implication in this equivalence.

THEOREM 4.1. *The graphs K_5 and $K_{3,3}$ are not planar.*

Proof. Let us suppose, to the contrary, that the graph K_5 is embedded in the sphere S^2. Since K_5 has 5 vertices and 10 edges, by Euler formula each of its maps in the sphere determines 7 regions, each of which has at least 3 edges on its boundary (K_5 has no loops or multiple edges). So, these 7 regions are bounded by at least 21 edges, where each edge is counted twice. We obtain $21 \leq 2e = 20$, which is a contradiction.

Let us suppose now, to the contrary, that the graph $K_{3,3}$ is embedded in the sphere S^2. Since the graph $K_{3,3}$ has 6 vertices and 9 edges, any of its maps in the sphere determines (by the Euler formula) 5 regions. As we have noticed before, any map of a bipartite graph determines regions with at least 4 edges. So, there are at least 20 edges on the boundary of these regions. Each edge is counted twice. So, we obtain $20 \leq 2e = 18$, which is a contradiction. ■

5. Coloring of maps

Consider now an embedding of a graph G in the surface S (a map on S). We want to color the regions of this map so that adjacent regions (sharing a common edge on their boundaries) are colored with different colors. We call such colorings *proper*. It is clear from the definition that such a coloring of a map induces a proper coloring of the vertices of the dual graph, where the adjacent vertices of the dual graph are colored with different colors. Also, if we have a proper coloring of the dual graph, we immediately obtain a proper coloring of the original map.

For each surface S (including the plane $\mathbb{R}^2$), we define $\kappa(S)$, the *coloring number* of S as the minimal number of colors needed to properly color any finite map on S. Similarly as for the maps themselves, the proper colorings of the maps in the plane and the maps in the sphere induce one each other, and so the coloring numbers of the plane and the sphere coincide. This enables us to reduce our attention (when dealing with colorings of maps on surfaces) to the case of closed surfaces.

If the graph G could be embedded in the surface S, and the dual graph contains the complete graph K_n as a subgraph, then the coloring number of S is at least n. Considering the map on the sphere obtained as the image of K_4 (1-skeleton of a tetrahedron), we see that the coloring number of the sphere (and the plane) is at least 4.

The problem of determining the coloring number of the plane asks (in everyday language) for the minimal number of colors needed for a proper coloring of the countries (which are required to be connected regions—so the countries like U.S.A. are not allowed!), in any possible geographic map on the Earth. This problem was raised by Guthrie already in 1852. The first false "proof" that 4 colors suffice was provided by Kempe in 1878. Heawood, in 1890, pointed to the error in this "proof", and showed that five colors suffice to color any map in the plane.

The conjecture that four colors suffice has become widely known as *the four-color conjecture*. It attracted a lot of attention, and became one of the best known and most important problems in mathematics in general for a long period of time. It was confirmed only after a lot of incorrect "proofs" and more than a century of unsuccessful attacks on the problem in 1976, by Appel and Haken, who used a computer to check a huge number of different cases. By the way, that was the first well-known case of the problem whose solution relied on the use of computer so heavily.

Of course, this proof is beyond the scope of this text. Here we present the proof of Heawood's five-color theorem instead. We start by a lemma.

LEMMA 5.1. *Every planar graph contains a vertex with at most* 5 *neighbors.*

Proof. Let G be a planar graph, and let us denote by G' the graph obtained from G by removing its loops and taking only one of the edges among two vertices with multiple edges. Obviously, the graph G' is also planar, and each vertex in G' has the same number of neighbors as in G. So, it suffices to prove that there is a vertex in G' with at most 5 neighbors.

Let us denote by v, e and r the numbers of vertices, edges and regions determined by some embedding of G' in the sphere S^2.

If we denote by $n(x)$ the number of neighbors of the vertex x, we have $n(x)$ edges starting from x, and so $\sum_{x\in V} n(x) = 2e$ (since each edge has 2 vertices). Suppose to the contrary that every vertex has at least 6 neighbors. Then we have $2e \geq 6v$, or $v \leq \frac{e}{3}$.

If we denote by $m(R)$ the number of edges on the boundary of the region R, we have $\sum_R m(R) = 2e$ (since each edge bounds 2 regions). Every region is bounded by at least 3 edges (since the graph G' has no loops or multiple edges), and so we obtain $2e \geq 3r$, or $r \leq \frac{2e}{3}$.

Substituting in the Euler formula, we have

$$2 = \chi(S^2) = v - e + r \leq \frac{e}{3} - e + \frac{2e}{3} = 0.$$

This contradiction proves the lemma. ∎

THEOREM 5.2. *Any map in the plane could be properly colored with* 5 *colors.*

Proof. We reformulate the problem in terms of graphs. Any map in the plane determines its dual graph which is planar. So, we have to prove that every planar graph could be properly colored by 5 colors. We prove this by induction on the number of vertices. For the graphs with at most 5 vertices the statement is trivial.

Suppose each planar graph with n vertices could be properly colored with 5 colors, and consider the planar graph G with $n+1$ vertices. By lemma, there is a vertex $x \in G$ with at most 5 neighbors. Consider the graph G' obtained from G by removing the vertex x and edges having x as a vertex. The graph G' is planar and it has n vertices, and so it could be properly colored with 5 colors by induction hypothesis.

If there are at most four neighbors of x in G, or if they are colored with at most 4 different colors, we could extend the coloring of G' to the one of G by assigning the fifth color to x.

Suppose now that neighbors of x are vertices x_1, x_2, x_3, x_4, x_5 in the cyclic order, and that they are colored with five different colors c_1, c_2, c_3, c_4, c_5 respectively. Consider now the subgraph $G_{1,3}$ of G' consisting of the vertices colored with c_1 and c_3, and of all edges of G' connecting these vertices. If x_1 and x_3 belong to the different connected components of $G_{1,3}$, we could reverse the coloring on the component containing x_1 and obtain new proper coloring of G'. But, no neighbor of x is colored with c_1 in this new coloring, and we could assign the color c_1 to the vertex x to obtain the proper coloring of G.

If the vertices x_1 and x_3 belong to the same component of $G_{1,3}$, there is a path $p_{1,3}$ between these two vertices containing only vertices colored with c_1 and c_3 and edges joining such vertices.

Now we consider the subgraph $G_{2,4}$ of G' consisting of the vertices colored with c_2 and c_4, and of all the edges of G' connecting these vertices. If x_2 and x_4 belong to the different connected components of $G_{2,4}$, we could, in the same way as before, obtain new proper coloring of G' which could be extended to the proper coloring of G.

If the vertices x_2 and x_4 belong to the same component of $G_{2,4}$, there is a path $p_{2,4}$ between these two vertices containing only vertices colored with c_2 and c_4 and edges joining such vertices. The path $p_{1,3}$ together with the edges connecting x to x_1, and x to x_3 respectively determines a loop bounding a curved disc which contains either the vertex x_2 or the vertex x_4, but not the both of them. So, it is obvious that the paths $p_{1,3}$ and $p_{2,4}$ have to intersect each other in the graph G, but this is impossible since their vertices are colored with different colors and so have to be different. (Notice that this is also a consequence of the non-planarity of the graph K_5.) This contradiction completes the proof. ∎

We could ask the same question for any closed surface S: What is the coloring number of S? It turns out, surprisingly, that the most difficult case (by far) is the case of the sphere, which we discussed above.

Heawood conjectured that for every closed surface S, its coloring number $\kappa(S)$, equals the following number $H(S)$, which is called the Heawood number of the surface S:

$$H(S) = \left[\frac{7 + \sqrt{49 - 24\chi(S)}}{2} \right].$$

Note that $H(S^2) = 4$, and $\kappa(S^2) = H(S^2)$ is the four-color theorem of Appel and Haken. In what follows we describe the answer to Heawood's conjecture in other cases.

First we prove the following.

Theorem 5.3. *For any closed surface S, $H(S)$ colors suffice to color any map on S, or $\kappa(S) \leq H(S)$.*

Proof. Let us denote $N = \kappa(S)$, and let $\varphi\colon G \to S$ be a map on S requiring N colors for its coloring. This means that its dual graph G^* could be colored by N, and could not be colored by $N-1$ colors (where different colors are assigned to neighboring vertices of G^*).

We could take a minimal such graph, with the respect to the operation of removing one vertex and the edges incident to it.

We prove that every vertex in this minimal graph G^* is incident to at least $N-1$ edges. Suppose to the contrary, that some vertex x is incident to at most $N-2$ edges. After removing the vertex x and all the edges incident to x from the graph G^*, we obtain the graph G' which could be colored by $N-1$ colors (since G^* was minimal). Since we assumed that x has at most $N-2$ neighbors, there is one of these $N-1$ colors not used for the coloring of any of its neighbors. Then, we could assign this color to x, and obtain in this way the coloring of G^* by $N-1$ colors, which is a contradiction.

So, every vertex of G^* is incident to at least $N-1$ edges, and we have $(N-1)v \leq 2e$, where again v, e and r denote the number of vertices, edges and regions determined by the graph G^* embedded in S.

Since a dual graph contains neither loops nor multiple edges, each region has at least 3 edges on its boundary, which implies $3r \leq 2e$, or $r \leq \frac{2e}{3}$.

The graph G^* need not to have regions homeomorphic to open discs. However, by the remark at the end of section 3, we have $\chi(S) \leq v - e + r$. So,

$$\chi(S) \leq v - e + r \leq v - \frac{e}{3},$$
$$(N-1)v \leq 2e \leq 6v - 6\chi(S),$$
$$N - 1 \leq 6 - \frac{6\chi(S)}{v}.$$

We have already discussed the case $S = S^2$. Let now $S = \mathbb{R}\mathrm{P}^2$. Then $\chi(S) = 1$, and the above inequality gives $N - 1 \leq 6 - \frac{6}{v} < 6$, which implies $N \leq 6 = H(\mathbb{R}\mathrm{P}^2)$.

Finally, for any other surface S, we have $\chi(S) \leq 0$. Since the coloring of G^* requires N colors, we certainly have $v \geq N$, and so

$$N - 1 \leq 6 - \frac{6\chi(S)}{N},$$

which implies $N^2 - 7N + 6\chi(S) \leq 0$. Therefore, the number N is contained between the roots of the quadratic equation $x^2 - 7x + 6\chi(S) = 0$. So, it is not greater than bigger of these roots, i.e.,

$$N \leq \frac{7 + \sqrt{49 - 24\chi(S)}}{2} = H(S). \quad \blacksquare$$

It remains to check whether $H(S)$ colors are necessary for coloring of the maps on the surface S. To show this to be the case, one needs to find a map on S, which requires that many colors.

In the case of the sphere S^2, a trivial example is provided by a map of four regions each two of which are neighbors. Its dual graph is the complete graph on four vertices. Below, we provide the examples in the cases of the torus T and the projective plane $\mathbb{RP}^2$, showing $\kappa(T) = 7$ and $\kappa(\mathbb{RP}^2) = 6$.

Exercise 5.4. *There is a map on the torus requiring* 7 *colors.*

We use the standard quotient model of the torus T obtained from the square in which the opposite sides are identified in pairs.

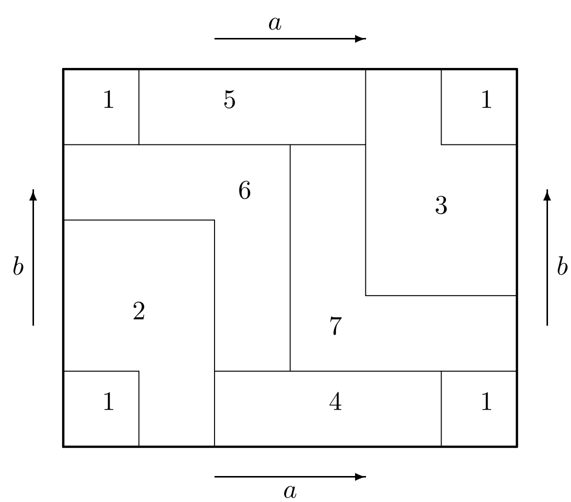

Fig. 2

Due to the described identification, the above map on T has 7 regions which are connected and each two of which are adjacent. So, its dual graph is the complete graph on 7 vertices K_7, and its coloring requires 7 colors. Remember, not even K_5 could have been embedded in S^2, and here we described the embedding of K_7 in T.

Exercise 5.5. *There is a map on the projective plane requiring* 6 *colors.*

We use the standard quotient model of $\mathbb{RP}^2$ obtained from the square in which the opposite boundary points are identified.

Check that the map on $\mathbb{RP}^2$ (Fig. 3) has 6 connected regions each two of which are adjacent. So, its dual graph is the complete graph on 6 vertices K_6, and its coloring requires 6 colors.

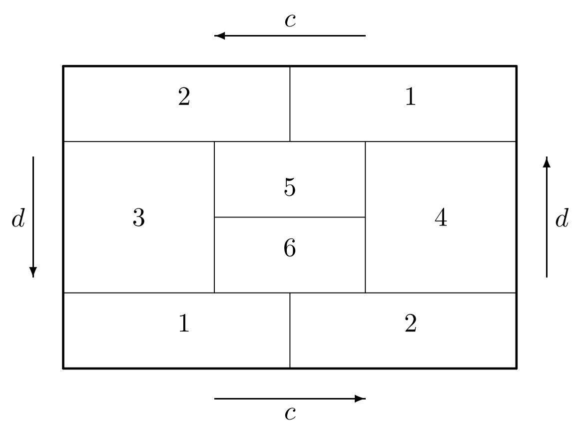

Fig. 3

The similar examples are constructed in all other cases, except for the case of the Klein bottle $K = N_2$, in which case it turned out that its coloring number is 6 rather than $H(K) = 7$. So, Heawood's conjecture is not true in the case of the Klein bottle and it is true in all other cases.

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University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Beograd, Serbia

E-mail: `vrecica@matf.bg.ac.rs`