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FARKAS' LEMMA OF ALTERNATIVE

Milojica Jaćimović

Dedicated to Professor Milosav Marjanović on the occasion of his 80th birthday

Abstract. We will present Farkas' formulation of the theorem of alternative related to solvability of a system of linear inequalities and present one review of the proofs based on quite different ideas.

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1. Introduction

Formulation of theorems in the form of alternative is widely used in mathematics. The following result related to system of linear equations

Either

(Ia) $$\exists x \textit{ such that } Ax = b$$

or

(IIa) $$\exists z \textit{ such that } A^\top z = 0 \textit{ and } \langle b, z\rangle \neq 0.$$

is a well-known example of a theorem of alternatives.

Especially, theorems about convergence of iterative processes are sometimes formulated as the theorems of alternatives. The problem of finding solutions of polynomial equations was considered in [6], and the statement was proved about convergence of a canonical sequences of iterative process associated with equation, in the form of alternative: *Either polynomial equation has no real roots or the canonical sequence converges and determines a root.*

The first theorem of alternative related to systems of linear equations and inequalities was published in 1873 by P. Gordon. Later, new theorems of alternative were proved, and they were widely used in proving of the existence of solutions in linear algebra and analysis, and in derivation of the necessary conditions for optimality. C.G. Broyden writes [2] (see also [5]) that "Theorems of alternatives lie at the heart of the mathematical programming." Farkas' theorem of alternative (known as Farkas's lemma), is present in university courses on optimization, either in its original form or as a duality theorem in linear programming. Let us note that

Farkas' lemma can be interpreted as a finite-dimensional version of Hahn-Banach theorem.

In this paper, in Section 2, we give five different proofs of the theorem of alternative. The proof connected with separation theorem is based on geometrical interpretation of the theorem and it is probably the shortest and the most popular. However, it cannot be considered as an "elementary proof", because some "topological arguments" (though simple) are used in it. In the second proof the so-called Fourier-Motzkin's method of elimination [3] is used which can be considered as a particular case of the well-known Tarski's theorem on quantifier elimination. This interesting method can in principle be used for construction of theorem provers, but the volume of computing grows too fast with dimension of the problem.

The proof by induction of the Farkas' lemma is infrequent in literature. Method o induction was used in [1] for proving of one generalization of Farkas' lemma. Here, we follow this proof.

The fourth proof which belongs to C.G. Broyden [2] is also algebraic. Namely, Broyden proved one property of orthogonal matrices from which Farkas's lemma can be derived.

Finally, let us note that Farkas's lemma has been applied to different problems in economics. We also present one such application, because it can contribute to understanding of this lemma.

2. Farkas' lemma

We will begin with formulation of one statement known as Farkas' lemma.

Teorema 1. (*Farkas' lemma*) *Let A be a matrix of order $m \times n$ and b a vector-column of dimension m. Then either*

$$\exists x \geq 0 \text{ such that } Ax = b \tag{I}$$

or

$$\exists z \text{ such that } A^\top z \leq 0 \text{ and } \langle b, z\rangle > 0. \tag{II}$$

It is clear that (II) is equivalent to

$$\exists z \text{ such that } A^\top z \geq 0 \text{ and } \langle b, z\rangle < 0. \tag{II'}$$

G. Farkas was a professor of Theoretical physics at the University of Kolozsvar (Hungary). He obtained this result while he was solving the problem of mechanical equilibrium posed by J. Fourier in 1798. He published it for the first time in 1898 in Hungarian, but Farkas' best-known exposition of his famous lemma was published in German in 1902.

If we denote the vectors-columns of matrix A by $a^1, a^2, \ldots, a^n$, we obtain an equivalent form of this theorem.

The inequality $\langle b, z\rangle \ge 0$ *is a consequence of the system of inequalities*

$$\langle a^1, z\rangle \ge 0,\ \langle a^2, z\rangle \ge 0,\ \ldots,\ \langle a^m, z\rangle \ge 0$$

if and only if vector b *is a linear combination*

$$b = x_1 a^1 + x_2 a^2 + \cdots + x_n a^n$$

with nonnegative coefficients $x_1, x_2, \ldots, x_n$.

Farkas's lemma shows that non-solvability of the system can be "certified": the solution z of (II) can be used as a "certificate" that the system (I) cannot be solved. Such certificates in mathematics are called *obstructions.*

There are several generalizations of Farkas' lemma. A part of these generalizations is related to the solvability of linear equation systems in other sets (besides $\mathbb{R}^n_+$). For example, the following theorem gives a criterion of solvability of linear equation systems in integers.

THEOREM 2. *Let* A *be a rational matrix and* b *a rational column vector. Then either*

system $Ax = b$ *has an integer solution,*

or

$\exists y$ *such that* $A^\top y$ *is an integer and* $\langle b, y\rangle$ *is not an integer.*

3. Proofs of Farkas' lemma

There are several variants of theorems of alternative and a lot of their proofs. Here, we will present some of the proofs of Farkas's lemma.

We will separately prove that the systems (I) and (II) are not solvable simultaneously. Assume the contrary, that there exist $x_0 \in R^n$ and $z_0 \in R^m$ that are solutions to (I) and (II) respectively. Then, we have

$$0 = \langle x_0, 0\rangle = \langle x_0, A^\top z_0\rangle = \langle Ax_0, z_0\rangle = \langle b, z_0\rangle > 0.$$

Thus, we arrive at a contradiction, and the first part of Farkas's lemma is proved.

Now, we will present some proofs of the second part of Farkas's lemma.

Proof 1. Let us suppose that the system (I) has no solution. Then $b \notin C := \{Ax : x \ge 0\}$. The set C is convex and closed, so by the separation theorem of closed convex sets, there exists a hyperplane $H := \{x : \langle z, x\rangle = \alpha\}$ containing b ($\alpha = \langle z, b\rangle$) such that

$$(\forall y \in C)\ \langle z, y\rangle < \alpha\ \Rightarrow\ (\forall x \in R^n_+)\ \langle A^\top z, x\rangle < \alpha.$$

This inequality is possible only for $\alpha > 0$ and $A^\top z \le 0$. So, we conclude that there exists $z \in R^m$, satisfying (II). ∎

Let us remark that in this very short and elegant geometric proof, the proof that the set $C := \{Ax : x \ge 0\}$ is closed was omitted. In addition, separation theorem is intuitively acceptable, but its proof is not so easy. These two facts are the most sensitive part of the geometrical proof of the theorem of alternative.

Note also that the existence of separating hyperplane for the set C and $b \notin C$ can be derived as a consequence of Farkas' lemma. It means that there is a ball $B(b,r)$ such that $C \cap B(b,r) = \emptyset$ and C is a closed set.

Instead of the separation theorem, one can use in this proof some properties of the projection on a convex closed set. Namely, if there is no $x \in R^n_+$ such that $Ax = b$, then there is a unique point $u = Ay = \pi_C(b)$, where $\pi_C(b)$ is the projection of b on C. The key property of the projection gives:

$$\langle b - Ay, Ax - Ay\rangle \le 0 \;\text{ for all } x \ge 0.$$

Let us put $z = b - u = -Ay + b$. Then,

$$\langle x - y, A^\top z\rangle \le 0, \quad \text{for all } x \ge 0.$$

From here, for $x = y + e_i \ge 0$, where $e_1, e_2, \dots, e_n$ is the standard basis of R^n, we have that the i-th component $(A^\top y)_i$ of $A^\top y$ is nonpositive, i.e. $A^\top z \le 0$. Besides,

$$\langle z, u\rangle = \langle z, Ay\rangle = \langle -Ay + b, Au\rangle = \langle b - Ay, Au - Ax\rangle \ge 0,$$

and

$$\langle b, z\rangle = \langle u + z, z\rangle = \langle u, z\rangle + \langle z, z\rangle > 0.$$

In [4], the proof of existence of the projection of b on C was given by introducing a simple iterative algorithm. Then the closedness of set C can be derived as an immediate consequence of the existence of projection of any point $b \in \infty R^n$ on C. Hence, Dax's proof can be considered as an indirect proof of the closedness of set C.

Proof 2. In this proof (see [3]) the so-called Fourier-Motzkin method for elimination of variables in linear inequalities will be used. This method can be considered as a particular case of Tarski's quantifier elimination theorem. It can be used for building of a theorem provers for this case. But, even in the case of a system of linear inequalities with only existential quantifiers, the method has very fast growth of the number of computational operation.

Denote by $a_1, a_2, \dots, a_m$ and $a^1, a^2, \dots, a^n$ the rows and the columns of matrix $A = (a_{ij})_{m\times n}$. Then, system (II) can be written in the form

$$\langle a^1, z\rangle \ge 0,\, \langle a^2, z\rangle \ge 0, \dots, \langle a^n, z\rangle \ge 0, \langle a^{n+1}, z\rangle \le -\beta < 0,$$

where vector b is denoted by a^{n+1}.

For example, suppose that we wish to eliminate the variable z_1 from the above system. Let us denote $I^+ = \{i : a_{1i} > 0\}$, $I^- = \{i : a_{1i} < 0\}$, $I^0 = \{i : a_{1i} = 0\}$. The new system of inequalities will be constructed using the following rules.

For each pair $(k,l) \in I^+ \times I^-$ let us multiply the inequalities $\langle a^k, z\rangle \le 0$ and $\langle a^l, z\rangle \le 0$ by $-a_{1l} > 0$ and $a_{1k} > 0$, respectively. Adding these two inequalities, we obtain a new one that does not contain the variable z_l. All inequalities obtained in this way will be added to those already in I_0. If I^+ (or I^-) is empty, we simply delete inequalities with indices in I^- (or in I^+). The inequalities with indices in I_0 form a new system of linear inequalities $Bz' \le d$, $z' = (z_2, \dots, z_n)$. The procedure of elimination of variable z_1 is described.

Let us remark that if $z' = (z'_2, \dots, z'_n)$ is a solution of the system $Bz' \le d$, and

$$\max_{l \in I^-} a_l 1^{-1}(-\sum_{j=2}^{n} a_{lj} + b_l) \le z_1 \le \min_{k \in I^+} a_k 1^{-1}(\sum_{j=2}^{n} a_{kj} - b_k)$$

then $z = (z_1, z') = (z_1, z_2, \dots, z_n)$ is a solution of the system (II).

Suppose that the system (II): $A^\top z \le 0$, $\langle -b, z\rangle \le -\beta < 0$ has no solution. Applying Fourier-Motzkin method for the elimination of the variables $z_1, z_2, \dots, z_n$, one obtains a system of inequalities without variables, that is a contradiction. This procedure converts the system (II) in inconsistent system

$$\begin{pmatrix} R & q \end{pmatrix} \begin{pmatrix} A^\top \\ -b^\top \end{pmatrix} = \begin{pmatrix} R & q \end{pmatrix} \begin{pmatrix} 0 & -\beta \end{pmatrix},$$

with $\beta > 0$ and nonnegative elements of the matrix $(R\ q)$. It means that $RA^\top - qb^t = 0$, where at least one $q_i \ne 0$. Consequently, there is $x \ge 0$ such that $Ax - b = 0$. ∎

Proof 3. The proof by induction is the third that will be presented here. In spite of its simplicity, this proof is quite infrequent in literature. We will expose this proof by following the proof of a generalization of Farkas' lemma from [1].

For $n = 1$ the systems (I) and (II) have the following form

$$\exists x \in \mathbb{R}_+ \text{ such that } ax = b \ (a, b \in \mathbb{R}^m$$
$$\exists z \in \mathbb{R}^m \text{ such that } \langle a, z\rangle \le 0 \text{ and } \langle b, z\rangle > 0.$$

and the statement is obviously valid.

Let us assume that the statement is valid for all natural n and let us consider m-dimensional vectors $a^1, a^2, \dots, a^n, a^{n+1}$. We have to prove that: (I) there exist nonnegative real numbers $x_1, x_2, \dots, x_n, x_{n+1}$ such that $b = x_1a^1 + \dots + x_na^n + x_{n+1}a^{n+1}$ or (II) there exists $z \in \mathbb{R}^m$ such that $\langle a^1, z\rangle \le 0, \dots, \langle a^n, z\rangle \le 0$, $\langle a^{n+1}, z\rangle \le 0$, $\langle b, z\rangle > 0$.

By induction hypothesis, we have that: (i) there exist nonnegative real numbers $x_1, x_2, \dots, x_n$ such that $b = x_1a^1 + \dots + x_na^n$ or (ii) there exists $z \in \mathbb{R}^m$ such that $\langle a^1, z\rangle \le 0, \dots, \langle a^n, z\rangle \le 0$, $\langle b, z\rangle > 0$. In case (i) it is enough to put $x_{n+1} = 0$ and we will obtain that (I) is valid. In case (ii), we will consider two possibilities: (ii-1) $\langle a^{n+1}, z\rangle \le 0$ and then we obtain that (II) is valid.

So, it remains to consider the case when there exists $z = \overline{z} \in \mathbb{R}^m$ such that $\langle a^1, \overline{z}\rangle \le 0, \dots, \langle a^n, \overline{z}\rangle \le 0$, $\langle b, \overline{z}\rangle > 0$ but $\langle a^{n+1}, \overline{z}\rangle > 0$. Let us consider two systems

(A) $$\langle a^1, z\rangle \le 0, \dots, \langle a^n, z\rangle \le 0, \langle a^{n+1}, z\rangle \le 0, \langle b, z\rangle > 0.$$

and

$$\text{(B)}\qquad \langle a^1, z\rangle \le 0,\ \dots,\ \langle a^n, z\rangle \le 0,\ \langle a^{n+1}, z\rangle = 0,\ \langle b, z\rangle > 0.$$

Each solution of (B)is a solution of (A). Further, if z is a solution of (A) then $z+\overline{z}$ is a solution of (B). Hence, (A) has a solution (in this case (II) is valid) if and only if (B) has a solution.

Now, let us consider vectors $c^i = a^i - \lambda_i a^{n+1}$, $i = 1, 2, \dots, n$ and $b' = b - \mu a^{n+1}$ where $\lambda_i = \frac{\langle a^i,\overline{z}\rangle}{\langle a^{n+1},\overline{z}\rangle} \le 0$ and $\mu = \frac{\langle b,\overline{z}\rangle}{\langle a^{n+1},\overline{z}\rangle} > 0$. It is easy to see that the system (B) has a solution if and only if system

$$\text{(C)}\qquad \langle c^1, z\rangle \le 0,\ \dots,\ \langle c^n, z\rangle \le 0,\ \langle a^{n+1}, z\rangle = 0,\ \langle b', z\rangle > 0.$$

has a solution. So, we have to concern the system (B).

Using the induction hypothesis again, we have that: (j) there exist nonnegative real numbers $y_1, y_2, \dots, y_n$ such that $b' = y_1c^1 + \dots + y_nc^n$ or (jj) there exists $u \in \mathbb{R}^m$ such that $\langle c^1, u\rangle \le 0,\ \dots,\ \langle c^n, u\rangle \le 0,\ \langle b', u\rangle > 0$. In case (jj), we have that the vector $z = u - \gamma\overline{z}$ where $\gamma = \frac{\langle b,u\rangle}{\langle a^{n+1},\overline{z}\rangle}$, is a solution of (C) and consequently (II) is valid.

If there exist nonnegative real numbers $y_1, y_2, \dots, y_n$ such that $b' = y_1c^1 + \dots + y_nc^n$ (case (jj)), then

$$b' = b - \mu a^{n+1} = y_1c^1 + \dots + y_nc^n = y_1a^1 - \lambda_1y_1a^{n+1} + \dots + y_na^n - \lambda_ny_na^{n+1}.$$

Hence,

$$b = y_1a^1 + \dots + y_na^n + y_{n+1}a^{n+1}$$

where $y_1, y_2, \dots, y_n$ and $y_{n+1} = \mu - \lambda_1y_1 - \dots - \lambda_ny_n \ge 0$. It maens that (I) is valid. ∎

The fourth proof appeared in G. Broyden's paper [2]. It is based on one property of orthogonal matrices that is referred as a Broyden's theorem.

Broyden's theorem. *If $Q = (q_{ij})_{n\times n}$ is an orthogonal matrix, then there exists a vector $x > 0$ and a unique diagonal matrix $S = diag(s_1, s_2, \dots, s_n)$ such that $s_i = \pm 1$ and $SQx = x$.*

The Broyden's proof of this theorem is by induction. For $m = 1$ the theorem is trivially true (Q and S are both equal to either $+1$ or -1). Assume the theorem is true for all orthogonal matrices of order $m\times m$. Let $Q = (q_{ij})_{(m+1)\times(m+1)}$ be an orthogonal matrix and let

$$Q = \begin{pmatrix} P & r \\ q^t & \alpha \end{pmatrix},$$

where $P = (p_{ij})_{m\times m}$. If $\alpha = 1$ or $\alpha = -1$, then $r = q = 0$ and the step induction becomes trivial. So, in what follows we can assume that $|\alpha| < 1$. Since Q is an orthogonal matrix,

$$P^\top P + q^\top q = I,\ P^\top r + \alpha q = 0,\ r^\top r + \alpha^2 = 1.$$

It follows from these equations that the matrices

$$Q_1 = P - \frac{rq^\top}{\alpha - 1},\ Q_1 = P - \frac{rq^\top}{\alpha + 1}$$

are orthogonal and that

$$Q_2^\top Q_1 = I - q\frac{2}{1-\alpha^2}q^\top.$$

Using induction assumption we conclude that there are $x_1 > 0$ and $x_2 > 0$ and diagonal sign matrices S_1 and S_2 such that $S_1Q_1x_1 = x_1$, $S_2Q_2x_2 = x_2$. From this we obtain that

$$\langle S_1x_1, S_2x_2\rangle = \langle Q_1x_1, Q_2x_2\rangle = \langle Q_2^\top Q_1x_1, x_2\rangle = \langle x_1, x_2\rangle - \frac{2}{1-\alpha^2}\langle x_1, q\rangle\langle x_2, q\rangle.$$

Now, we have to consider two cases.

Case 1. If $S_1 \neq S_2$, then $\langle S_1x_1, S_2x_2\rangle < \langle x_1, x_2\rangle$. So, $\langle q, x_1\rangle \neq 0$ and $\langle q, x_2\rangle \neq 0$ and both scalar products have the some signs. For

$$\eta_1 = \frac{-\langle q, x_1\rangle}{\alpha - 1},\ \eta_2 = \frac{-\langle q, x_2\rangle}{\alpha + 1},\ z_1 = \begin{pmatrix} x_1 \\ \eta_1 \end{pmatrix},\ z_2 = \begin{pmatrix} x_2 \\ -\eta_2 \end{pmatrix},$$
$$S^1 = \begin{pmatrix} S_1 & 0 \\ 0 & 1 \end{pmatrix},\ S^2 = \begin{pmatrix} S_2 & 0 \\ 0 & -1 \end{pmatrix},$$

we have

$$Qz_1 = S^1z_1,\ Qz_2 = S^2z_2.$$

Now, since $|\alpha| < 1$, and both scalar products $\langle q, x_i\rangle$ have the same signs, one of η_i is positive and one of the vectors z_1 and z_2 is the required vector. In case 1 the proof is completed.

Case 2. If $S_1 = S_2$, then $\langle S_1x_1, S_2x_2\rangle = \langle x_1, x_2\rangle$ and at least one of $\langle q, x_1\rangle$ and $\langle q, x_2\rangle$ is zero. We will assume that $\langle q, x_1\rangle \neq 0$ and $\langle q, x_2\rangle = 0$. So $Px_1 = S_1x_1 = Q_1x_1$ and $Qz_1 = S^1z_1$, where

$$z_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix},\ S^1 = \begin{pmatrix} S_1 & 0 \\ 0 & \sigma \end{pmatrix},\ \sigma = \pm 1 \text{ may be chosen arbitrarily.}$$

Now, if we rewrite Q in the form

$$Q = \begin{pmatrix} \alpha_1 & q_1^\top \\ r_1 & P_1 \end{pmatrix}$$

where P_1 is a matrix of the type $m \times m$, and repeat the previous argument, we will obtain that there exist a positive vector x_2 and a diagonal matrix S_2 with ± 1 on diagonal, such that

$$Qz_2 = S^2z_2,\ z_2 = \begin{pmatrix} 0 \\ x_2 \end{pmatrix},\ S^2 = \begin{pmatrix} 0 & \sigma' \\ S_1 & 0 \end{pmatrix},$$

where $\sigma' = \pm 1,\ \sigma = \pm 1$ may be chosen arbitrarily.

Combining the equation $Qz_1 = S^1 z_1$ and $Qz_2 = S^2 z_2$, we obtain

$$Q(z_1 + z_2) = S^1 z_1 + S^2 z_2,$$

with strictly positive coordinate z_{1_j} and z_{2_j} for $j \geq 2$. If for some $j \in \{2, \dots, m\}$ the corresponding diagonal elements of S^1 and S^2 are different, then $\|S^1 z_1 + S^2 z_2\| < \|z_1 + z_2\| = \|Q(z_1 + z_2)\|$, but it is a contradiction with the previous equality. We can choose the elements σ and σ' so that $S^1 = S^2$. So, we have $Q(z_1 + z_2) = S^1(z_1 + z_2)$, and since $z_1 + z_2$ is strictly positive and S^1 is a diagonal matrix with ± 1 on diagonal, this establishes the induction in case $S_1 = S_2$.

Assume that there exist two positive vectors x and y and two diagonal matrices S and R with ± 1 on diagonal, such that $Qx = Sx = Ry$, and $S \neq R$. Then

$$\langle x, y\rangle = \langle Qx, Qy\rangle = \langle Sx, Ry\rangle < \langle x, y\rangle,$$

giving a contradiction. Therefore, $R = S$. This completes the proof of Broyden's theorem. ■

The next result known as Tucker's theorem is a simple consequence of Broyden's theorem.

Tucker's theorem. *Let A be a skew-symmetric matrix. Then there exists $y \geq 0$ such that $Ay \geq 0$ and $y + Ay > 0$.*

Proof. Since A is skew-symmetric then $(I + A)^{-1}(I - A)$ is orthogonal, so that there exist a positive vector x and a unique matrix S such that

$$(I - A)^{-1}(I + A)x = Sx \Leftrightarrow x + Ax = Sx - ASx.$$

If we denote $y = x + Sx$, $z = Ay = Ax + ASx = x - Sx$, then every coordinate y_j of vector y is equal either $2x_j$ or zero, so $y \geq 0$. Similarly, $z \geq 0$. But $y + z = y + Ay = 2x > 0$. ■

Proof 4. (Broyden's proof of Farkas's lema) Apply Tucker's theorem to skew-symmetric matrix

$$B = \begin{pmatrix} O & O & A & -b \\ O & O & -A & b \\ -A^\top & A^\top & O & 0 \\ b^\top & -b^\top & O^\top & 0. \end{pmatrix}$$

By Tucker's theorem, there exists a positive vector $y = (z_1, z_2, x, t)^\top$ such that $y + By > 0$. Consider the two cases: $t > 0$ and $t = 0$ with $z = z_1 - z_2$. If $t > 0$ the vector y may be normalized so that $t = 1$, from which we obtain $Ax = b$. If $t = 0$ then $A^\top z \leq 0$ and $\langle b, z\rangle > 0$. ■

In his paper Broyden discussed the question of relation between Tucker's theorem, Farkas's lemma and Broyden's theorem. He derived Tucker's theorem and Farkas's lemma as simple consequences of Broyden's theorem. In [7] Ross and Terlaky showed that Broyden theorem is also a simple consequence of Tucker's theorem and Farkas's lemma. It means that Farkas's lemma, Tucker's theorem and Broyden's theorem are equivalent results.

In [7], the following observation was made: for a given orthogonal matrix Q the existence of a unique diagonal matrix S with diagonal elements $s_{ii} = \pm 1$ and positive vector x such that $Qx = Sx$ is equivalent to the existence of a positive vector x such that

$$|Qx| = x,\, x > 0 \Leftrightarrow -x \leq Qx \leq x,\, x > 0,$$

where $|y|$ denotes the vector whose entries are absolute values of the entries of vector y. So, if one finds a vector x satisfying the previous conditions, then $s_{ii} = 1$ if $x_i > 0$ and $s_{ii} = -1$ if $x_i < 0$.

This Roos' and Terlaky's comment was a very perspicuous answer on Broyden's remark [2] that "However, it may yet be possible to construct such algorithm (for determining of the sign of a matrix) and the author suspects that if this is the case then any successful example would have more than a passing resemblance to the interior point algorithms, but only the passage of time will resolve this conjecture".

3. An example of economic interpretation of Farkas' lemma

Let us consider a market in which m different assets are traded. Suppose that there are n possible states of market at the end of the period. Let us denote by $p_1, p_2, \ldots, p_m$ the given prices at the beginning of the period. Let $A = (a_{ij})_{m\times n}$ be a payoff matrix, where a_{ij} denote the price of i-th asset at the end of the period, if the market is in state j. A portfolio of assets is then a vector $x = (y_1, y_2, \ldots, y_m)$, where y_i denotes the amount of i-th asset. If the component y_i is positive, then the investor which bought y_i units of i-th asset will receive $a_{ji}y_i$ if j-th state materializes. But, one supposes that sell short position is allowed, that means selling some quantity of i-th asset at the beginning of the period and buying it back at the end. Consequently, y_i can be negative. In this case one must pay out $a_{ji}|y_i|$.

At the end of the period the investor with portfolio $y = (y_1, y_2, \ldots, y_)$ will receive $z = A^\top y$, where $z_j = \sum_{i=1}^n a_{ji}y_i$ is a result from portfolio $y = (y_1, y_2, \ldots, y_n)$ in the state j. The cost of acquiring a portfolio x is given by $\langle p, y\rangle$. In asset pricing the main problem is to determine the prices p_i. In arbitrage pricing theory, the standard condition is the absence of arbitrage: there is no portfolio with a negative cost but a positive return in every state. This principle can be formulated as

if $A^\top y \geq 0$ *then* $\langle p, y\rangle \geq 0$

or as

there is no y *such that* $A^\top y \geq 0,\ \langle p, y\rangle < 0.$

It means that only certain prices p are consistent with principle of absence of arbitrage. Here we have new question: how to describe these prices? The answer is contained in the following variant of Farkas' lemma.

Theorem 3. *The absence of arbitrage condition holds if and only if there is a nonnegative vector* $q \in \mathbb{R}^n_+$ *such that the prices of assets are given by* $p_i = \sum_{j=1}^m a_{ji}q_j$.

Let us comment the above theorem. If the absence of arbitrage condition holds, then the system $A^y \geq 0$, $\langle p, y\rangle < 0$ has no solutions. Now, from Farkas' lemma we derive the existence of vector $q \in \mathbb{R}^n$, such that $p = Aq$. Then for $p^* = \frac{p}{\sum_i q_i}$ and $q^* = \frac{q}{\sum_i q_i}$ we obtain the equality in scaled prices: $p^* = Aq^*$. Further, vector q^* can be interpreted as a vector of distribution of probabilities. Hence, there is a distribution of probabilities under which the expected return of every asset is equal to its price. These probabilities are called risk-neutral probabilities. Hence, Farkas' lemma can be formulated as: *the existence of risk-neutral probabilities is a consequence of the absence of arbitrage condition.*

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Department of Mathematics, University of Montenegro, Podgorica