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IMPROPER INTEGRAL

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Abstract. In the following we shall show how to extend the class of Riemann integrable functions with the class of functions integrable in the improper sense. The extension is made in such a way that the same procedure can be applied to the class of Lebesgue integrable functions.

1. Regular and singular points

In this section by integrability we mean the Riemann integrability.

1. Let $[a,b]$ be an interval from the extended set of real numbers $\overline{\mathbf{R}}$ and let f be a function mapping $[a,b]$ into the set of reals $\mathbf{R}$. A point $c \in [a,b]$ is a regular point of the function f if the function f is integrable in some neighborhood of c. Otherwise c is a singular point of the function f.

Theorem 1. *The function f is integrable on the interval $[a,b]$ if and only if each point from $[a,b]$ is a regular point of the function f.*

Proof. If the function f is integrable on the interval $[a,b]$, then, according to the definition of a regular point above, each point from the interval $[a,b]$ is regular. Suppose each point from $[a,b]$ is regular point of the function f. Let U_x be the neighborhood of the point $x \in [a,b]$ (in the topology of the interval $[a,b]$) on which f is integrable. Because of the compactness of the interval $[a,b]$ its cover $\{U_x \mid x \in [a,b]\}$ contains a finite subcover $\{U_x \mid x \in \{x_1, x_2, \dots, x_n\}\}$. Since f is integrable on each of the intervals U_x, $x \in [a,b]$, and since $[a,b]$ can be represented as a union of the finite subfamily of those intervals, then f is integrable on $[a,b]$. ∎

2. Examples.

1. If $a = -\infty$, then a is a singular point of the function f.

2. If $b = +\infty$, then b is a singular point of the function f.

3. If f is unbounded in each neighborhood of the point $c \in [a,b] \cap \mathbf{R}$, then c is a singular point of f.

The preceding three statements follow immediately from the facts that the Riemann integral can be defined only on the finite intervals and that a Riemann integrable function is necessarily bounded. If the function f has only finitely many singular points, then each of them belongs to one of the three types described in the previous examples. This fact is given by the following

Theorem 2. *If c is a finite isolated singular point of the function f, then f is unbounded in each neighborhood of c.*

Proof. We shall consider the case when $c = b$ and when b is the only singular point of f.

Suppose f is bounded in some neighborhood of the point b. On the complement of that neighborhood it is integrable and therefore bounded. It follows that f is bounded on the interval $[a,b]$. Therefore there exists such a real number M, that $|f(t)| \le M$ for each $t \in [a,b]$. Let $\epsilon > 0$. In the interval (a,b) there exists such a number y, that $b-y < \epsilon/6M$. Since f is integrable on $[a,y]$, there exists such $\delta > 0$, that $|\sigma - \sigma'| < \epsilon/3$ for each two integral sums σ, σ' of the function f corresponding to the partitions of the interval $[a,y]$ with diameters less than δ. We can suppose that $\delta < \epsilon/12M$. Let

$$P : a = t_0 < t_1 < t_2 < \cdots < t_m = b, \qquad P' : a = t'_0 < t'_1 < t'_2 < \cdots < t'_m = b$$

be two partitions of the interval $[a,b]$ with diameters less than δ and choose one point in every subinterval of each partition:

$$\tau_i \in [t_i, t_{i+1}],\ i = 0,1,\ldots,m-1, \qquad \tau'_j \in [t'_j, t'_{j+1}],\ j = 0,1,\ldots,n-1.$$

It is enough to prove that the absolute value of the difference of two integral sums

$$\Sigma = \sum_{i=0}^{m-1} f(\tau_i)\Delta t_i \quad \text{and} \quad \Sigma' = \sum_{j=0}^{n-1} f(\tau'_j)\Delta t'_j$$

is less than ϵ. Suppose the point y belongs to the interval $[t_k, t_{k+1}]$ of the partition P and to the interval $[t'_l, t'_{l+1}]$ of the partition P'. Let $\overline{P}$ and $\overline{P}'$ be partitions obtained from partitions P and P' by adding the point y and let $\overline{\Sigma}$ and $\overline{\Sigma}'$ be the corresponding integral sums:

$$\overline{\Sigma} = \sum_{i=0}^{k-1} f(\tau_i)\Delta t_i + f(y)[y - t_k] + f(y)[t_{k+1} - y] + \sum_{i=k+1}^{m-1} f(\tau_i)\Delta t_i,$$

$$\overline{\Sigma}' = \sum_{j=0}^{l-1} f(\tau'_j)\Delta t'_j + f(y)[y - t'_l] + f(y)[t'_{l+1} - y] + \sum_{j=l+1}^{n-1} f(\tau'_j)\Delta t'_j.$$

Sums

$$\sigma = \sum_{i=0}^{k-1} f(\tau_i)\Delta t_i + f(y)[y - t_k], \quad \sigma' = \sum_{j=0}^{l-1} f(\tau'_j)\Delta t'_j + f(y)[y - t'_l]$$

are integral sums of the function f corresponding to the partitions of the interval $[a,y]$ with diameters less than δ. Therefore $|\sigma - \sigma'| < \epsilon/3$. The following estimates are valid:

$$|\Sigma - \overline{\Sigma}| = |f(\tau_k) - f(y)|(t_{k+1} - t_k) \le 2M\delta \le \frac{\epsilon}{6},$$

$$|\Sigma' - \overline{\Sigma}'| = |f(\tau'_l) - f(y)|(t'_{l+1} - t'_l) \le 2M\delta \le \frac{\epsilon}{6},$$

$$|\overline{\Sigma} - \sigma| = |f(y)[t_{k+1} - y] + \sum_{i=k+1}^{m-1} f(\tau_i)\Delta t_i| \le M(b-y) \le \frac{\epsilon}{6},$$

$$|\overline{\Sigma}' - \sigma'| = |f(y)[t'_{l+1} - y] + \sum_{j=l+1}^{n-1} f(\tau'_j)\Delta t'_j| \le M(b-y) \le \frac{\epsilon}{6}.$$

It follows that

$$\begin{aligned} |\Sigma - \Sigma'| &\le |\Sigma - \overline{\Sigma}| + |\overline{\Sigma} - \sigma| + |\sigma - \sigma'| + |\sigma' - \overline{\Sigma}'| + |\overline{\Sigma} -' \Sigma'| \\ &\le \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \le \epsilon. \quad \blacksquare \end{aligned}$$

2. Definition of the improper integral

1. Let f be a function mapping the interval $[a,b]$ of the extended real line $\overline{\mathbf{R}}$ in the set of real numbers. If a is the only singular point of the function f and if there exists the finite

$$\lim_{x\to a} \int_x^b f(t)\,dt,$$

or if b is the only singular point of the function f and if there exists the finite

$$\lim_{x\to b} \int_a^y f(t)\,dt,$$

or if a and b are the only singular points of the function f and if there exists the finite

$$\lim_{\substack{x\to a\\ y\to b}} \int_x^y f(t)\,dt,$$

we say that the function f has the improper integral on the interval $[a,b]$. The limit in all three cases we shall denote by

$$\int_a^b f(t)\,dt$$

and we shall call it the improper integral of the function f on the interval $[a,b]$.

2. Suppose the function f, mapping the interval $[a,b]$ of the extended real line $\overline{\mathbf{R}}$ in the set of reals, has finitely many singular points from which at least one belongs to the interior of the interval $[a,b]$. Let $c_1 < c_2 < \cdots < c_k$ be singular points of the function f lying in the interior of the interval $[a,b]$. If f has proper or improper integral in each of the intervals $[a,c_1]$, $[c_1,c_2]$, ... , $[c_k,b]$, we shall say that it has the improper integral on the interval $[a,b]$. The value of the improper integral of the function f on the interval $[a,b]$ is defined by

$$\int_a^b f(t)\,dt = \int_a^{c_1} f(t)\,dt + \int_{c_1}^{c_2} f(t)\,dt + \cdots + \int_{c_k}^b f(t)\,dt.$$

3. Basic properties of the improper integral

In this section by integrability we mean the integrability in the proper or in the improper sense.

1. Let f be a function defined on the interval $[a,b]$ and let c be the interior point of that interval. The function f is integrable on the interval $[a,b]$ if and only if it is integrable on the intervals $[a,c]$ and $[c,b]$. If it is integrable then the following equality holds

$$(*) \qquad \int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

As in the case of the proper integral, in the case of the improper integral we define

$$\int_a^b f(t)\,dt = 0, \quad \text{if} \quad a = b,$$
$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt, \quad \text{if} \quad a > b.$$

The equality $(*)$ will hold for arbitrarily distributed points a, b and c under the condition that there exist all three integrals appearing in that equality.

2. If functions f and g are integrable on the interval $[a,b]$, then $f+g$ is also integrable on $[a,b]$ and

$$\int_a^b [f(t)+g(t)]\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$$

Using the preceding statement it is not difficult to prove that by changing the values of the integrable function in finitely many points we obtain integrable function whose integral is equal to the integral of the initial function. This is why we can speak about the integrability and about the integral of a function on some interval even in the case when that function is not defined in finitely many points of that interval.

3. If f is integrable on $[a,b]$ and if λ is a real number, then the function λf is integrable on the interval $[a,b]$ and

$$\int_a^b \lambda f(t)\,dt = \lambda \int_a^b f(t)\,dt.$$

4. If functions f and g are integrable on the interval $[a,b]$ and if $f(t) \le g(t)$ for each $t \in [a,b]$, then

$$\int_a^b f(t)\,dt \le \int_a^b g(t)\,dt.$$

5. If the function f has finitely many singular points in the interval $[a,b]$ and if $|f|$ is integrable then f is integrable and

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

Remark. It is not difficult to prove that each regular point of the function f must be the regular point of the function $|f|$, too. Therefore the assumption that the function f has finitely many singular points implies that the function $|f|$ has finitely many singular points, too.

6. Let f be a integrable function on the interval $[a, b]$. Then the function F defined by

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on the interval $[a, b]$ and differentiable in each continuity point of f.

7. (Newton-Leibniz formula) Suppose the function $F\colon [a, b] \to \mathbf{R}$ is continuous and has integrable derivative $F' = f$. Then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

8. (Change of variable) Let $f\colon [a, b] \to \mathbf{R}$ be an integrable function, let $g\colon [\alpha, \beta] \to \overline{\mathbf{R}}$ be monotone, continuous with derivative having only finitely many singular points. If $g(\alpha) = a$ and $g(\beta) = b$, then

$$\int_a^b f(t)\,dt = \int_\alpha^\beta f(g(u))g'(u)\,du.$$

It is not difficult to prove the preceding eight statements by using the corresponding statements for proper integrals and the definition of the improper integral. We leave these proofs to the reader.

4. Commentaries

1. The class of Lebesgue integrable functions can be extended in the similar way by the class of functions integrable in the improper sense. The notion of the singular point can be introduced in the similar way as in the first section. The Theorem 1 remains valid, while Theorem 2 and examples are valid only in the case of the Riemann integral. The definition of the improper integral from the second section remains unchanged. The first six properties of the integral enlisted in the third section hold also for the extension of the Lebesgue integral. The remaining two properties hold under modified assumptions, similar to those appearing in the corresponding statements for the proper Lebesgue integral.

2. The initial class of functions could be the class of Riemann or Lebesgue integrable functions completed with the corresponding class of functions integrable in the improper sense. In such a way a new extension of the class of integrable functions will be obtained. That procedure of extending can be repeated arbitrarily many times.

Example. Let $f\colon [0, +\infty) \to \mathbf{R}$ be defined by $f(t) = 1/\sqrt{t \sin t}$. The function f has infinitely many singular points (the points of the form $k\pi$, k$= 0, 1, 2, \ldots,$ and $+\infty$), and therefore its improper integral does not exist. However, on every finite interval $[0, y]$, the function f is integrable in the improper sense and in that case there exists finite

$$\lim_{y \to +\infty} \int_0^y \frac{dt}{\sqrt{t \sin t}}.$$

3. Theorem 2 could be proved more easily using Lebesgue criterion for Riemann integrability. However, in some areas of mathematics we use the Riemann integral of functions having Banach space as a co-domain. One should bear this fact in mind also in the case when developing theory of Riemann integral on the class of real functions. All definitions, theorems and proofs should be formulated in such a way that they can be applied without change to the case of vector valued functions (this, of course, in the case when the corresponding theorems are valid in the case of vector valued functions). That is the reason why we have chosen the given proof of the Theorem 2.

4. Integration by parts has not been mentioned among the properties of the integral in the third section. The proposition about the integration by parts can be formulated as follows:

Suppose continuous functions f and g, defined on the interval $[a,b]$, have integrable derivatives. Then

$$\int_a^b f(t)g'(t)\,dt = f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t)\,dt.$$

The proposition formulated in such a way has the disadvantage that in most of the cases the application of integration by parts can not be justified. This is well illustrated by the following example:

$$\int_0^1 \ln t\,dt = t\ln t\,\Big|_0^1 - \int_0^1 dt = 0 - 0 - 1 = -1.$$

If we wished to formulate the proposition about the integration by parts which could be applicable to the above and to similar cases, its formulation would be rather cumbersome. That is why we shall not formulate it here. If there is a need for integration by parts in the improper integral in some problem, then it could be solved by first calculating the indefinite integral and then by applying Newton-Leibniz formula.

5. In this article we studied the definition and properties of the improper integral. The conditions for integrability for improper integrals as well as improper integrals depending on parameters were not considered here.

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