THE TEACHING OF MATHEMATICS
2008, Vol. XI, 2, pp. 93–96

SOLVING SEXTICS BY DIVISION METHOD

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Abstract. We describe a method for solving reducible sextics over the real field, which makes use of a simple division. A procedure to synthesize such sextics is given and a numerical example is solved to extract the roots of a sextic with the proposed method.

ZDM Subject Classification: H24; *AMS Subject Classification*: 00A35.

Key words and phrases: Sextics, reducible polynomials, reducible sextic.

1. Introduction

A polynomial is said to be reducible over a given field if it can be factored into polynomials of lower degree with coefficients in that field; otherwise it is termed as an irreducible polynomial [1]. This paper describes a simple division method to decompose a reducible sextic over the real field into a product of two polynomial factors, one quadratic and one quartic. The conditions on the coefficients of such reducible sextic are derived.

2. The division method

Consider the following sextic polynomial reducible over the real field:

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \tag{1}$$

where the coefficients, a_0, a_1, a_2, a_3, a_4 and a_5 are rational. Let the reducible sextic, $p(x)$, be such that the sum of its two roots is equal to the sum of its remaining four roots. Thus if the roots of $p(x)$ are x_1, x_2, x_3, x_4, x_5 and x_6, then they satisfy the relation

$$x_1 + x_2 = x_3 + x_4 + x_5 + x_6. \tag{2}$$

However since we know that the sum of all the six roots is:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = -a_5 \tag{3}$$

it then follows from (2) that

$$x_1 + x_2 = -\frac{a_5}{2}. \tag{4}$$

The quadratic polynomial factor of the sextic, p(x), containing the roots x_1 and x_2, can be expressed as:

$$q(x) = x^2 + \frac{a_5}{2}x + n, \tag{5}$$

where $n = x_1x_2$. Note that n is an unknown to be determined. We now divide $p(x)$ by $q(x)$ as shown below.

$$\frac{p(x)}{q(x)} = x^4 + \frac{a_5}{2}x^3 + \left(a_4 - n - \frac{a_5^2}{4}\right)x^2 + \left(a_3 - \frac{a_4a_5}{2} + \frac{a_5^2}{8}\right)x + b + \frac{\left(a_1 - na_3 + \frac{na_4a_5}{2} - \frac{na_5^3}{8} - \frac{a_5b}{2}\right)x + a_0 - nb}{x^2 + \frac{a_5}{2}x + n}, \tag{6}$$

where b in the above expression is given by:

$$b = a_2 - na_4 + n^2 + \frac{na_5^2}{4} - \frac{a_3a_5}{2} + \frac{a_4a_5^2}{4} - \frac{a_5^4}{16}. \tag{7}$$

Since $q(x)$ is a factor of $p(x)$, the remainder term in (6) should be zero for all values of x, which means each of the terms, $a_1 - na_3 + \frac{na_4a_5}{2} - \frac{na_5^3}{8} - \frac{a_5b}{2}$ and $a_0 - nb$ should vanish as shown below:

$$a_1 - na_3 + \frac{na_4a_5}{2} - \frac{na_5^3}{8} - \frac{a_5b}{2} = 0 \tag{8}$$

$$a_0 - nb = 0. \tag{9}$$

From (9) an expression for b is obtained as $b = \frac{a_0}{n}$, which is used to eliminate b from (7) and (8) resulting in the following cubic and quadratic equations in n respectively.

$$n^3 + \frac{a_5^2 - 4a_4}{4}n^2 + \frac{16a_2 + 4a_4a_5^2 - 8a_3a_5 - a_5^4}{16}n - a_0 = 0, \tag{10}$$

$$n^2 + \frac{8a_1}{4a_4a_5 - 8a_3 - a_5^3}n - \frac{4a_0a_5}{4a_4a_5 - 8a_3 - a_5^2} = 0. \tag{11}$$

From the quadratic (11) we obtain two values for n as shown below.

$$n = \frac{4a_1}{4a_4a_5 - 8a_3 - a_5^2}\left[-1 \pm \sqrt{1 + \frac{a_0a_5(4a_4a_5 - 8a_3 - a_5^3)}{4a_1^2}}\right]. \tag{12}$$

However the desired value of n is the one, which is real and satisfies the cubic (10). Thus the desired value of n is the common real root of (10) and (11). Picking this value of n, and using the expressions (5) and (6), the sextic polynomial $p(x)$ is now expressed as the product of quadratic and quartic factors as shown below.

$$p(x) = \left(x^2 + \frac{a_5}{2}x + n\right) \times \left[x^4 + \frac{a_5}{2}x^3 + \left(a_4 - n - \frac{a_5^2}{4}\right)x^2 + \left(a_3 - \frac{a_4a_5}{2} + \frac{a_5^3}{8}\right)x + \frac{a_0}{n}\right]. \tag{13}$$

The roots of $p(x)$ are obtained by equating each of the two factors in (13) to zero and solving the resulting quadratic and quartic equations. When the quadratic factor $q(x)$ is equated to zero and solved, the two roots of $p(x)$, x_1 and x_2, are determined as shown below

$$x_1 = \frac{-a_5 + \sqrt{a_5^2 - 16n}}{4}, \qquad x_2 = \frac{-a_5 - \sqrt{a_5^2 - 16n}}{4}. \tag{14}$$

The remaining four roots of $p(x)$, x_3, x_4, x_5, and x_6, are determined when the quartic factor of $p(x)$ is equated to zero as shown below

$$x^4 + \frac{a_5}{2}x^3 + \left(a_4 - n - \frac{a_5^2}{4}\right)x^2 + \left(a_3 - \frac{a_4 a_5}{2} + \frac{a_5^3}{8}\right)x + \frac{a_0}{n} = 0, \tag{15}$$

and solved by the method available in literature [2]. Thus all the six roots of sextic polynomial, $p(x)$, are determined.

3. Synthesis of sextic polynomials

We now describe a procedure to synthesize such sextic polynomials. Notice that, the two expressions for n [given by (12)] do not contain the coefficient a_2. Therefore the remaining rational coefficients, a_0, a_1, a_3, a_4, and a_5, are chosen such that the values of n obtained from (12) are real. The cubic equation (10) is rearranged such that it becomes an expression for a_2 as indicated below.

$$a_2 = \frac{16a_0 + (a_5^4 + 8a_3a_5 - 4a_4a_5^2)n + (16a_4 - 4a_5^2)n^2 - 16n^3}{16n}. \tag{16}$$

The two real values of n [obtained from (12)] are then used in (16) to obtain two values of a_2, which in turn yield two sextic polynomials. Notice that the expressions (12) and (16) serve as the conditions for the coefficients of $p(x)$ to satisfy, so that $p(x)$ is decomposable with this method.

4. Numerical example

Let us synthesize the sextic polynomial with the coefficients, a_0, a_1, a_3, a_4, and a_5, chosen as shown below.

$$p(x) = x^6 + 2x^5 + 3x^4 + x^3 + a_2x^2 + 4x + 12.$$

The synthesis implies determination of the remaining coefficient a_2. To start with the two values of n are determined using the expressions given in (12) as: 2 and -6. We then choose $n = 2$ and determine a_2 from the expression (16) as $a_2 = 5$. Thus the sextic polynomial synthesized is given by:

$$p(x) = x^6 + 2x^5 + 3x^4 + x^3 + 5x^2 + 4x + 12.$$

The above sextic can be decomposed using (13) as shown below.

$$p(x) = (x^2 + x + 2)(x^4 + x^3 - x + 6).$$

The two roots of the sextic (which are the roots of the quadratic factor) are determined from the expressions given in (14) as

$$x_1 = \frac{1 + i\sqrt{7}}{2}, \qquad x_2 = \frac{1 - i\sqrt{7}}{2}.$$

The remaining four roots are obtained by solving the quartic, $x^4 + x^3 - x + 6 = 0$, with the method given in [2] as: $x_3 = -1.410229 + 1.160678\,i$, $x_4 = -1.410229 + 1.160678\,i$, $x_5 = 0.910229 + 0.984932\,i$, and $x_6 = 0.910229 - 0.984932\,i$.

5. Conclusions

A method to decompose a reducible sextic over the real field into a product of quadratic and quartic factors is given, which uses simple division. The sextic in the proposed method is such that, the sum of its two roots is equal to the sum of its remaining four roots. The conditions on the coefficients of such sextics are derived and a procedure to synthesize these polynomials is described.

Acknowledgements. The author thanks the management of Bharat Electronics for supporting this work.

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