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CONVEXITY OF THE INVERSE FUNCTION

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Abstract. This note answers the following question: Having an invertible convex real valued function $f\colon A \to \mathbf{R}$, what can be said about convexity of f^{-1}?

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The intention of this author is to sketch a research theme for students of special mathematical schools (for example, mathematical gymnasia). The conditions under which two functions f and f^{-1} are convex (concave) in the same time are not found in the current books on calculus. Thus, a teacher should let her/his students get acquainted with all involved concepts leaving the question of these conditions open.

It is well known that if an invertible function f is increasing (decreasing), its inverse is of the same type. The question arises: what can be said about convexity of f and f^{-1}?

Probably we would think first of the exponential function $y = e^x$ and the function $y = x^2$ and conclude that convexity of one of the functions f or f^{-1} implies concavity of the other. But examples of the functions such that f and f^{-1} are both convex (concave) exist in abundance, one of them being $y = \dfrac{1}{x}$ which is inverse to itself! So, the next question is: Is there any rule?

Recall that a real function f defined on an interval $A \subset \mathbf{R}$ is convex on A if for each $x_1, x_2 \in A$ and $\alpha_1, \alpha_2 \in [0, 1]$ such that $\alpha_1 + \alpha_2 = 1$

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 f(x_1) + \alpha_2 f(x_2) \tag{1}$$

holds.

If in (1) the inequality $<$ (respectively $\geqslant$, $>$) takes place, the function is strictly convex (respectively concave, strictly concave).

The above convexity condition is equivalent to (see [1]): For every three points $x_1, x, x_2 \in A$ such that $x_1 < x < x_2$

$$\frac{f(x) - f(x_1)}{x - x_1} \leqslant \frac{f(x_2) - f(x)}{x_2 - x} \tag{2}$$

is true.

A well-known fact is that a two times differentiable function f on an interval is convex (concave) if and only if its second derivative f'' is nonnegative (nonpositive). (See [1] or [3].) Thus we easily prove the next statement.

Proposition 1. *Let* $f\colon (a,b) \overset{\text{onto}}{\to} (c,d) \subset \mathbf{R}$ *be two times differentiable function,* $f^{-1}\colon (c,d) \to \mathbf{R}$ *be its inverse, and let* $f'(x) \neq 0$.

(1) *If* f *and* f^{-1} *are decreasing functions, convexity of one of them implies convexity of the other.*

(2) *If* f *and* f^{-1} *are increasing functions, convexity of one of them implies concavity of the other.*

Proof. For $x = f^{-1}(y)$ the following holds: $(f^{-1})'(y) = \dfrac{1}{f'(x)}$ and

$$(f^{-1})''(y) = \left(\frac{1}{f'(x)}\right)'(y) = \frac{d}{dx}\left(\frac{1}{f'(x)}\right)x'(y) = \frac{-f''(x)}{f'(x)^2}x'(y) = \frac{-f''(x)}{(f'(x))^2}\cdot\frac{1}{f'(x)}$$

where $x = f^{-1}(y)$.

(1) If f is a decreasing function then $f'(x) < 0$, so $(f^{-1})''(y)$ and $f''(x)$ have the same sign.

(2) If f is an increasing function then $f'(x) > 0$, hence $(f^{-1})''(y)$ and $f''(x)$ have opposite signs. ■

To prove the general case we use the following statement concerning continuity of convex functions (see [2, Chapter 1, Section 4.3], or [4, Tvrdjenje 2.3]).

Theorem 1. *Let* $f\colon (a,b) \to \mathbf{R}$ *be a convex function. Then*

1. f *is continuous on* (a,b);
2. *at each point* $x \in (a,b)$ *there exist the left-hand derivative* $f'_-(x)$ *and the right-hand derivative* $f'_+(x)$;
3. *the set of points in which* f *is not differentiable is at most countable.*

Proof. First we prove 2. Then, being continuous from the left and the right at each point x, the function f is continuous on (a,b).

Let $x, t \in (a,b)$, $x \neq t$. Denote by $\nu(x;t) = \dfrac{f(x)-f(t)}{x-t}$ the slope of the line segment passing through the points $(x, f(x))$, and $(t, f(t))$. Clearly, $\nu(x;t) = \nu(t;x)$. The condition (2) can be rewritten as

$$(2') \qquad \nu(x;x_1) \leqslant \nu(x;x_2).$$

Note that $\dfrac{f(x)-f(x_1)}{x-x_1} \leqslant \dfrac{f(x_2)-f(x)}{x_2-x}$ is equivalent to

$$\frac{f(x)-f(x_1)}{x-x_1} \leqslant \frac{(f(x)-f(x_1))+(f(x_2)-f(x))}{(x-x_1)+(x_2-x)} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

and

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{(f(x)-f(x_1))+(f(x_2)-f(x))}{(x-x_1)+(x_2-x)} \leqslant \frac{f(x_2)-f(x)}{x_2-x}.$$

By renumbering the points we get $\nu(x;t_1) \leqslant \nu(x;t_2)$ for $x < t_1 < t_2$ and $\nu(x;t_1) \leqslant \nu(x;t_2)$ for $t_1 < t_2 < x$. Thus, for a fixed $x \in (a,b)$ the slope function

$\nu(x;t) \equiv \varphi(t)$ is an increasing function. There exist $\lim_{t\to x-} \nu(x;t) = f'_-(x)$ and $\lim_{t\to x+} \nu(x;t) = f'_+(x)$. Moreover, $f'_-(x) \leqslant f'_+(x)$ by $(2')$.

3. The statement is proved in the standard way using the fact that the set $\mathbf{Q}$ of rational numbers is dense in $\mathbf{R}$. ■

Remark. That a convex function $f\colon [a,b] \to \mathbf{R}$ need not be continuous at the end points can be seen from the following example. Let $f(x) = x^2$ for $-1 < x < 1$, and $f(x) = 2$ for $x \in \{-1, 1\}$.

Proposition 2. *Let* $f\colon (a,b) \overset{\text{onto}}{\to} (c,d) \subset \mathbf{R}$ *be a convex function and let* $f^{-1}\colon (c,d) \to \mathbf{R}$ *be its inverse.*

(1) *If f is increasing then f^{-1} is increasing and concave.*

(2) *If f is decreasing then f^{-1} is decreasing and convex.*

Proof. Being continuous and invertible on (a,b) the function f is strictly monotone (as well as its inverse) and convex. Let $c < y_1 < y < y_2 < d$ and let x, x_1 and x_2 be the unique points from (a,b) for which $f(x) = y$, $f(x_i) = y_i, i = 1,2$ hold.

(1) If f is increasing then $x_1 < x < x_2$ and from $\dfrac{y-y_1}{x-x_1} \leqslant \dfrac{y_2-y}{x_2-x}$ it follows that $\dfrac{x-x_1}{y-y_1} \geqslant \dfrac{x_2-x}{y_2-y}$. Hence f^{-1} is concave.

(2) If f is decreasing then $x_1 > x > x_2$, so from $\dfrac{y-y_2}{x-x_2} \leqslant \dfrac{y_1-y}{x_1-x}$ it follows $\dfrac{y_2-y}{x-x_2} \geqslant \dfrac{y-y_1}{x_1-x}(> 0)$, hence $\dfrac{x-x_2}{y_2-y} \leqslant \dfrac{x_1-x}{y-y_1}$, i.e. $\dfrac{x_2-x}{y_2-y} \geqslant \dfrac{x-x_1}{y-y_1}$ and f^{-1} is a convex function. ■

Proposition 3. *Let* $f\colon [a,b] \overset{\text{onto}}{\to} A \subset \mathbf{R}$ *be a convex and invertible function and let f^{-1} be its inverse.*

(1) *If f is increasing then f^{-1} is concave on each interval $I \subset A$.*

(2) *If f is decreasing then f^{-1} is convex on each interval $I \subset A$.*

Proof. Let $f_1 = f|(a,b)$. The restriction f_1 is monotone on (a,b). Let $c = \lim_{x\to a+} f(x)$ and $d = \lim_{x\to b-} f(x)$. It must be $f(a) \geqslant c$ and $f(b) \geqslant d$.

If f is continuous at a or b, then for its inverse function on the interval $f([a,b))$, respectively $f((a,b])$, Proposition 2 holds.

If $f(a), f(b) > \max\{c,d\}$, then for f^{-1} Proposition 2 holds on the interval $f(a,b)$.

(1) If f is increasing on (a,b), and $f(a) = d$, then f^{-1} is concave on $(c,d]$.

(2) Dually, if f is decreasing on (a,b), and $f(b) = c$, then f^{-1} is convex on $(d,c]$. ■

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